The Cohomology of G-spaces (G compact group)

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P. A. Smith, around 1940, proved some striking new theorems about the group $\mathbb{Z}/p$ (p prime) acting on spheres and euclidean spaces. For example he showed:

Theorem. *If the group $G = (\mathbb{Z}/p)^r$ acts freely on the sphere S^n, then $r = 1$.*

An obvious question was then:

Rank Conjecture. *If G as above acts freely on a product of k spheres, is r less than or equal to k?*

This was proved for two spheres by A. Heller in 1959, and in 1980, G. Carlsson proved the result when the spheres were all of the same dimension. In 2010, B. Hanke proved it when $p > 3$ ($\dim X$).

We give a quick proof of the Smith and Carlsson theorems, but this proof uses much about the actual topology of the spheres, its manifold structure, etc., while the proofs of Smith and Carlsson used only the finite dimension of the space and its cohomology ring structure.

For a space constructed from G spheres in an inductive fashion by G fibrations, an easy proof of the Rank Conjecture is possible using an inductive argument, but this is a very special situation, as can be shown by counterexamples.

The goal of these lectures will be to use a more abstract, homotopy theoretic approach to these questions, so that one may use methods in this more flexible setting and achieve inductive arguments, similar to on the number spheres, to get more results.

Given a free action of G (any group) on a space X, the cohomology of the quotient space X/G is a module over the cohomology of the group $H^*(G)$. For $G = (Z/p)^r$, the second cohomology group $H^2(G)$ is isomorphic to G^*, the dual of G, and the polynomial ring on $H^2(G)$ is the central part of $H^*(G)$. In particular, $H^*(G)$ is nilpotent over this subalgebra.

For the free action of G on a finite dimension space, $H^*(G)$ acts nilpotently on $H^*(X/G)$, since X/G is also finite dimensional. On the other hand, if $H^*(X)$ is finitely generated, and $H^*(X/G)$ a nilpotent H*(G) module, one can show that $H^*(X/G)$ is also finitely generated.

The notion of $H^*(G)$ nilpotence becomes the central hypothesis in our analyses, and has several advantages:

(a) it may apply to infinite dimensional spaces, which may come up in our homotopy theoretic constructions; and
(b) it pulls back under G maps.

Many other properties of this condition will be described.

From now on we will assume that the action of G on the cohomology of X is the the trivial action (the identity). Some specific results:

Suppose that X has the homotopy type of a product of spheres each of which is of dimension either m or n.

Theorem. *If $m = 1$, then the Rank Conjecture holds.*

Theorem. *If $p = 2$, and $m = 3$ and n is odd, the Rank Conjecture holds.*

For odd p, a similar theorem holds with more complicated assumptions on n.