Topology of the Grünbaum hyperplane mass partition problem

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Let $\mathcal{M} = \{\mu_1, \dots, \mu_j\}$ be a collection of *masses*, i.e., finite Borel measures on $\mathbb{R}^d$ such that every affine hyperplane has measure zero, and let $\mathcal{H} = \{H_1, \dots, H_k\}$ be an arrangement of k hyperplanes in $\mathbb{R}^d$. Then $\mathcal{H}$ *equiparts* the collection of masses $\mathcal{M}$ if it cuts each of the j masses into 2^k orthants of equal measure, that is, if

$$\mu_\ell(\mathcal{O}_g^{\mathcal{H}}) = \tfrac{1}{2^k}\mu_\ell(\mathbb{R}^d)$$

for $1 \le \ell \le j$ and $g \in (\mathbb{Z}/2)^k$, where $\mathcal{O}_g^{\mathcal{H}}$ will be our notation for the orthants determined by $\mathcal{H}$. In his paper from 1960 Grünbaum suggested the following general measure partition problem:

Problem. *Determine the minimal dimension $d = \Delta(j,k)$ such that for every collection $\mathcal{M}$ of j masses in $\mathbb{R}^d$ there exists an arrangement of k hyperplanes $\mathcal{H}$ that equiparts $\mathcal{M}$.*

The Grünbaum hyperplane mass partition problem was extensively studied over more than a half century by many authors. It was a playground for testing the strengths of various different topological methods. The absence of complete answers and presence of number of claims with incomplete or even invalid proofs makes this problem into one of most challenging problems of Topological Combinatorics.

In this talk we review known results on the Grünbaum hyperplane mass partition problem, present new proofs for known results, explain gaps in proofs of number of claims and finally establish new values of $\Delta(j,k)$.