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Discrete Geometry 1 – Problem Sheet 1

Please hand in your solutions to Prof. Ziegler on **Wednesday, Oct. 23, 2013** before the lecture begins. Please put your name and student ID (if you have one) on the first page of your solutions and staple the sheets together.

Problem 1: *Number of Convex Sets* (3 + 3 + 3 Points)

For $d \geq 1$ we introduce the following equivalence relation for convex subsets A, B of $\mathbb{R}^d$:

$$A \sim B \;:\Longleftrightarrow\; \text{there exists an } x \in \mathbb{R}^d \text{ and an } \alpha > 0 \text{ such that } B = x + \alpha A.$$

Here $x + \alpha A$ denotes the set $\{x + \alpha y : y \in A\}$.

(a) How many distinct equivalence classes are there in the case $d = 1$? Describe the classes.

(b) In the case where $d = 1$, how many distinct equivalence classes are there, if the only restriction on α is $\alpha \neq 0$? Again, give a description of the classes.

(c) What is the cardinality of the set of equivalence classes for $d = 2$?

Problem 2: *Carathéodory's Theorem* (6 Points)

Prove the following statement known as Carathéodory's Theorem:

Let $S \subseteq \mathbb{R}^d$ be a set and $x \in \operatorname{conv}(S)$ be a point in the convex hull of S. Then there exists a set $R \subseteq S$ of cardinality at most $d+1$ such that $x \in \operatorname{conv}(R)$.

Hint: First show that every point $x \in \operatorname{conv}(S)$ can be written as a finite *convex combination* $x = \sum_{i=1}^{n} \lambda_i x_i$ for some $n \in \mathbb{N}$, where $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and $x_i \in S$. Then argue similarly as in the proof of Radon's Theorem[1].

[1] Radon's Theorem was incorrectly called Carathéodory's Lemma in the lecture.

Problem 3: *Cone Generated by a Set* (2 + 2 + 1 Points)

For any pair of distinct points $a, b \in \mathbb{R}^d$ let $[a, b[$ denote the closed halfline with endpoint a that passes through b. In other words

$$[a, b[\,:= \left\{ x \in \mathbb{R}^d : x = a + \alpha(b - a) \text{ for some } \alpha \geq 0 \right\}.$$

For any set $K \subseteq \mathbb{R}^d$ and any element $a \in \mathbb{R}^d$ we define the *cone* generated by K with *apex* a as follows:

$$\operatorname{cone}_a(K) \coloneqq \bigcup_{b \in K,\, b \neq a} [a, b[\,.$$

Let $K \subseteq \mathbb{R}^d$ be a convex set.

(a) Show that the set $\operatorname{cone}_a(K)$ is convex.

(b) Show that if K is open, then the set $\operatorname{cone}_a(K) \setminus \{a\}$ is open.

(c) Is statement (a) true for an arbitrary set $K \subseteq \mathbb{R}^d$?