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Discrete Geometry 1 – Problem Sheet 2

Please hand in your solutions to Prof. Ziegler on **Wednesday, Oct. 30, 2013** before the lecture begins. Please put your name and student ID (if you have one) on the first page of your solutions and staple the sheets together.

Problem 1: *Product and Minkowski Sum* (2 + 2 + 2 (+2) + 2 Points)

Let P_1 and P_2 be two polytopes of dimensions d_1 respectively d_2.

(a) Show that the Cartesian product $P_1 \times P_2$ is a polytope. What is the dimension of $P_1 \times P_2$?

(b) Prove that for every non-empty face F of $P_1 \times P_2$ there are unique faces $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$ such that $F = F_1 \times F_2$.

(c) Assume P_1 and P_2 are both polytopes in $\mathbb{R}^d$ for some $d \geq 0$. Show that the *Minkowski sum*

$$P_1 + P_2 := \{p_1 + p_2 : p_1 \in P_1 \text{ and } p_2 \in P_2\}$$

of P_1 and P_2 is a polytope. *Bonus:* Prove that if the Minkowski sum of two convex sets $K_1, K_2 \subseteq \mathbb{R}^d$ is a polytope, then both K_1 and K_2 are polytopes.

(d) Show that if F is a non-empty face of $P_1 + P_2$, there are faces $F_i \subseteq P_i$ such that $F = F_1 + F_2$ and that the choice of F_1 and F_2 is unique.

Problem 2: *Crosspolytope* (3 + 3 Points)

For $d \geq 1$ the d-dimensional *crosspolytope* is given by

$$C_d^{\triangle} := \operatorname{conv}\{\pm e_1, \pm e_2, \dots, \pm e_d\}.$$

Here e_i denotes the i-th standard basis vector of $\mathbb{R}^d$.

(a) Let $u, v \in \{\pm e_1, \pm e_2, \dots, \pm e_d\}$ be given such that $u \neq \pm v$. Show that the interval $[u, v] = \operatorname{conv}\{u, v\}$ is an edge of $C_d^{\triangle}$.

(b) Let $P = \operatorname{conv}(V)$ be a polytope and V its set of vertices. We call P *centrally symmetric* if $-P = P$. Show that a polytope P is centrally symmetric if and only if P is the image under a linear map of the n-dimensional crosspolytope $C_n^{\triangle}$ with $n = \frac{1}{2}|V|$.

Problem 3: *Adjacent Vertices – Tetrahedron* (6 Points)

Let P be a polytope. Recall that vertices of P are 0-dimensional faces of P and edges of P are 1-dimensional faces of P. Two distinct vertices x and y of P are *adjacent* if there is an edge e of P that has x and y as faces.

Let P now be 3-dimensional and assume that every two distinct vertices of P are adjacent. Show that P is a tetrahedron, that is, a 3-dimensional simplex.

Hint: Adroitly apply Radon's Theorem.