

## Discrete Geometry 1 – Problem Sheet 11

Please hand in your solutions to Prof. Ziegler on **Wednesday, Jan. 22, 2014** before the lecture begins.

**Problem 1:** *The Number of Faces: Extremal Properties* (4(+3) Points)

Fix  $d$  and let  $M_i(n)$  be the maximal and  $m_i(n)$  be the minimal number of  $i$ -faces for a simplicial  $d$ -polytope with  $n$  vertices.

- (a) (i) Show that  $M_i(n)$  and  $m_i(n)$  are polynomials in  $n$ .  
(ii) Compute the degrees of  $M_i(n)$  and  $m_i(n)$ .  
(iii) Compute the leading coefficients of  $M_i(n)$  and  $m_i(n)$ .
- (b) *Bonus:* What can be said if the polytope is not required to be simplicial?

**Problem 2:** *M-Sequences in Dimension 4 and 5* (4 Points)

Show that  $(1, a, b) \in \mathbb{Z}^3$  is an  $M$ -sequence if and only if

$$a, b \geq 0 \quad \text{and} \quad b \leq \binom{a+1}{2}.$$

**Problem 3:** *f-Vectors of Simplicial Polytopes* (6 Points)

Consider the following potential  $f$ -vectors of 4-polytopes:

$$v_1 = (1, 17, 57, 80, 40)$$

$$v_2 = (1, 13, 84, 142, 71)$$

$$v_3 = (1, 17, 100, 166, 83)$$

- (a) Compute the corresponding  $h$ - and  $g$ -vectors.  
(b) Which of the  $v_i$  is the  $f$ -vector of a simplicial 4-polytope?

**Problem 4:** *Counterexample for Motzkin's Unimodality Conjecture* (6 Points)

In the 1950s, T. S. Motzkin and others conjectured that  $f$ -vectors of convex polytopes are unimodal. Recall the definition of unimodality from Problem Sheet 6. A first counterexample was given by A. Björner in 1981 [1]. He constructed a simplicial polytope of dimension 24 with  $f_{14} > f_{15} < f_{16}$ .

In this exercise you will construct another counterexample: a simplicial polytope  $P$  of dimension 30 whose  $f$ -vector is not unimodal. You are encouraged to use a computer algebra program such as *Maple* or *Mathematica* to do your calculations. Let  $d = 30, n = 47$  and  $r = 65555$ . Define

$$g_1 := n - d - 1 + r,$$
$$g_k := \binom{n - d - 2 + k}{k}, \quad \text{for } k \geq 2.$$

- (a) Check that  $g = (1, g_1, \dots, g_{15})$  is the  $g$ -vector of a simplicial polytope  $P$ .
- (b) Compute the  $f$ -vector of  $P$  and show that it is *not* unimodal.
- (c) Describe a polytope with this  $f$ -vector.

## References

- [1] A. BJÖRNER, *The unimodality conjecture for convex polytopes*, Bulletin (New Series) of the American Mathematical Society, 4 (1981), pp. 187–188.