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Discrete Geometry 1

Solutions for the first exercise of Sheet 11

Problem 1: *The Number of Faces: Extremal Properties* (4(+3) Points)

Fix d and let $M_i(n)$ be the maximal and $m_i(n)$ be the minimal number of i-faces for a simplicial d-polytope with n vertices.

(a) (i) Show that $M_i(n)$ and $m_i(n)$ are polynomials in n.

The upper and lower bound theorem gives us that the f-vector of a simplicial d-polytope on n vertices is bounded by the stacked d-polytope on n vertices and the cyclic d-polytope on n vertices which are themselves simplicial. Thus

$$m_i(n) = f_i(\mathrm{Stack}_d(n) =) \begin{cases} \binom{d+1}{i+1} + (n-d-1)\binom{d}{i} & \text{for } i < d-1 \\ (d+1) + (n-d-1)(d-1) & \text{for } i = d-1 \end{cases}$$

$$M_i(n) = f_i(C_d(n)) = \frac{n-\delta(n-i-2)}{n-i-1} \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \binom{n-1-j}{i+1-j} \binom{n-(i+1)}{2j-(i+1)+\delta}$$

Those functions are obviously polynomials in n.

(ii) Compute the degrees of $M_i(n)$ and $m_i(n)$.

$$\deg m_i(n) = 1$$

In order to compute $\deg M_i(n)$, note that $\binom{m}{k} := 0$ if $k < 0$ or $k > m$. Using and $\lfloor \frac{d}{2} \rfloor \geq j$ and $n \geq d+1$ we deduce $n-(i+1) \geq 2j-(i+1)+\delta$ and with $n \geq d+1$ and $d \geq i+1$, we get $n-1-j \geq i+1-j$, i.e. $m \geq k$ is satisfied. If we want k to be non-negative we need to choose

$j \leq i+1$ and $j \geq \frac{i+1-\delta}{2}$. In that case the degree of each summand is $i+1-j+2j-(i+1)+\delta = j+\delta$ which maximizes for $j \in \min(i+1, \lfloor \frac{d}{2} \rfloor)$. Also considering the contribution of the factor $\frac{n-\delta(n-i-2)}{n-i-1}$, we get

$$\deg M_i(n) = -\delta + \min(i+1, \lfloor \tfrac{d}{2} \rfloor) + \delta = \min(i+1, \lfloor \tfrac{d}{2} \rfloor).$$

(iii) Compute the leading coefficients of $M_i(n)$ and $m_i(n)$.

The leading coefficients for $m_i(n)$ are $\binom{d}{i}$ for $i < d-1$ and $d-1$ for $i = d-1$.

For $M_i(n)$ the sum contributes with $\frac{1}{(i+1-k)!(2k-(i+1)+\delta)!}$ for $k = \min(i+1, \lfloor \frac{d}{2} \rfloor)$ whereas $\frac{n-\delta(n-i-2)}{n-i-1}$ contributes with $(i+2)^\delta$. Therefore the leading coefficient is

$$\frac{(i+2)^\delta}{(i+1-k)!(2k-(i+1)+\delta)!}.$$

For $k = i+1$ (i.e. $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor - 1$) this yields

$$\frac{(i+2)^\delta}{(i+1-(i+1))!(2(i+1)-(i+1)))!(2(i+1)-i)^\delta} = \frac{1}{(i+1)!}$$

For $k = \lfloor \frac{d}{2} \rfloor$ (i.e. $i \geq \lfloor \frac{d}{2} \rfloor$) we obtain

$$\begin{aligned} \frac{(i+2)^\delta}{(i+1-\lfloor \frac{d}{2} \rfloor)!(2\lfloor \frac{d}{2} \rfloor - (i+1)+\delta)!} &= \frac{(i+2)^\delta}{(i+1-\lfloor \frac{d}{2} \rfloor)!(d-(i+1))!} \\ &= \frac{(i+2)^\delta}{\lfloor \frac{d}{2} \rfloor !} \binom{\lfloor \frac{d}{2} \rfloor}{d-(i+1)} \end{aligned}$$

Another approach to analyze the f-vector of the cyclic polytope would be to use the fact that $C_d(n)$ is neighborly. This means we know roughly half of the entries of the face vector already.

$$f_i(C_d(n)) = \binom{n}{i+1} \quad \text{for } i < \lfloor \tfrac{d}{2} \rfloor.$$

Then use Dehn-Sommerville and the formulae to compute the h-vector from the f-vector and vice versa and argue why the functions are polynomials, deduce the degree and the leading coefficients

$$\begin{cases} \binom{\frac{d}{2}}{i+1-\frac{d}{2}} \frac{1}{\frac{d}{2}!} & \text{for even } d \\ \left(\binom{\lceil \frac{d}{2} \rceil}{i+1-\lfloor \frac{d}{2} \rfloor} + \binom{\lfloor \frac{d}{2} \rfloor}{i+1-\lceil \frac{d}{2} \rceil} \right) \frac{1}{\lfloor \frac{d}{2} \rfloor !} & \text{for odd } d. \end{cases}$$

Simple calculations show that the two formulae for the leading coefficient of $M_i(n)$ are actually identical.

(b) *Bonus:* What can be said if the polytope is not required to be simplicial?

Sketch: In that case $M_i(n)$ is still given by the cyclic polytope but the lower bound theorem doesn't apply anymore. For fixed d the polar of $C_d(n)$ gives an infinite family of counterexamples.