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Discrete Geometry 1 – Problem Sheet 4

Please hand in your solutions to Prof. Ziegler on **Wednesday, Nov. 13, 2013** before the lecture begins. Please put your name and student ID (if you have one) on the first page of your solutions and staple the sheets together.

Problem 1: *Transportation Polytopes* (3+3 Points)

Let $m, n \geq 1$ and $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ be given such that $a_i, b_j > 0$ and $\Delta = \sum_{i=1}^m a_i = \sum_{j=1}^n b_j$. We call

$$\begin{aligned} P = P(m,n;a,b) \coloneqq \Big\{(x_{ij})_{ij} \in \mathbb{R}^{m\times n} \ : & \, x_{ij} \geq 0 \text{ for } 1 \leq i \leq m,\ 1 \leq j \leq n, \\ & \sum_{i=1}^m x_{ij} = b_j \text{ for } 1 \leq j \leq n, \\ & \sum_{j=1}^n x_{ij} = a_i \text{ for } 1 \leq i \leq m\Big\} \end{aligned}$$

a *transportation polytope*.

(a) What is the dimension of $P(m,n;a,b)$?

(b) Let $m = n$ and $a = b = (1,\dots,1)$. How many vertices and how many facets does this $P(m,n;a,b)$ have? Give an explanation.

(c) Describe a point that lies in $P(m,n;a,b)$.
Hint: Show that $x_{ij} = \frac{1}{n^2}$ yields a point for the special case of part (b). Then generalize to the case where $a_i = \frac{1}{n}$ and $b_j = \frac{1}{m}$. Then generalize further.

Problem 2: *Combinatorial Isomorphisms* (2+2(+2) Points)

Two polytopes P_1, P_2 are called *combinatorially isomorphic* if their face lattices are isomorphic.

(a) Find two simplicial 3-polytopes with 6 vertices each that are not combinatorially isomorphic.

(b) Find two combinatorially non-isomorphic 3-polytopes P_1 and P_2 with the same sets of vertex figures. To be more precise, there must exist a bijection $\iota\colon V_1 \to V_2$ between the vertex sets of P_1 and P_2 such that the vertex figure of v is combinatorially isomorphic to the vertex figure of $\iota(v)$, for all $v \in V_1$.

(c) *Bonus:* Find two combinatorially non-isomorphic 4-polytopes P_1 and P_2 with the same sets of vertex figures.

Problem 3: *Minkowski–Weyl Representation Theorem* (4+3+3 Points)

For $d \geq 1$ let

$$C_d = [-1,1]^d \text{ be the } d\text{-dimensional unit cube and}$$
$$C_d^* = \operatorname{conv}\{\pm e_1, \ldots, \pm e_d\} \text{ be the } d\text{-dimensional crosspolytope.}$$

For $0 \leq n \leq d$ let f_n and f_n^* denote the number of n-dimensional faces of C_d and of C_d^*, respectively.

(a) Verify the Minkowski–Weyl Representation Theorem for the polytopes C_3 and C_3^*. In other words, calculate $\mathcal{V}$- and $\mathcal{H}$-representations for both polytopes.

(b) Describe the k-dimensional faces of C_d and calculate $f_0, \ldots, f_d$.

(c) Describe the k-dimensional faces of C_d^* and calculate $f_0^*, \ldots, f_d^*$.