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Discrete Geometry 1 – Problem Sheet 5

Please hand in your solutions to Prof. Ziegler on **Wednesday, Nov. 20, 2013** before the lecture begins. Please put your name and student ID (if you have one) on the first page of your solutions and staple the sheets together.

Problem 1: *Vertices and Facets of a Direct Sum* (4+3+1(+2) Points)

Let $\Delta_d \subset \mathbb{R}^d$ denote a d-simplex with the origin in its interior. The *direct sum* or *free sum* of Δ_d with itself is defined as

$$\Delta_d \oplus \Delta_d := \operatorname{conv}\left(\left\{(p,0) \in \mathbb{R}^{2d} : p \in \Delta_d\right\} \cup \left\{(0,p') \in \mathbb{R}^{2d} : p' \in \Delta_d\right\}\right) \subset \mathbb{R}^{2d}.$$

(a) Is $\Delta_d \oplus \Delta_d$ a polytope? How many vertices and how many facets does $\Delta_d \oplus \Delta_d$ have? What is its dimension?

(b) State the vertex-facet incidence Matrix $I = I(\Delta_d \oplus \Delta_d)$ of $\Delta_d \oplus \Delta_d$.

(c) Which combinatorial properties does $\Delta_d \oplus \Delta_d$ have that can be immediately determined from I?

(d) *Bonus:* What does I^t describe? Explain.

Problem 2: *Polarity* (4+2 Points)

(a) Compute explicitly and draw the polars of the following rectangles in the plane:
   - (i) R_1 with vertices $(0,0), (M,0), (M,1)$ and $(0,1)$, for $M > 0$ large.
   - (ii) R_2 with vertices $(-\varepsilon,-\varepsilon), (M,-\varepsilon), (M,1)$ and $(-\varepsilon,1)$, for $M > 0$ large and $\varepsilon > 0$ small.
   - (iii) R_3 with vertices $(\varepsilon,\varepsilon), (M,\varepsilon), (M,1)$ and $(\varepsilon,1)$, for $M > 0$ large and $\varepsilon > 0$ small.

(b) What happens in (ii) and (iii) if $\varepsilon \to 0$ or $M \to \infty$? Give an explanation.

Problem 3: *A vertex-facet incidence matrix* (6 Points)

Is this the vertex-facet incidence matrix of a convex polytope? If yes, how many vertices and how many facets does it have? What is its dimension? What else can you say about it? Can you draw it?

$$I = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4: Bonus Problem: *Farkas Lemma* ((6) Points)

Given a vector $x \in \mathbb{R}^d$, the relation $x \geq 0$ means that all components x_i for $i = 1, \ldots, d$ are non-negative. Consider the following version of Farkas' Lemma:

Let $A \in \mathbb{R}^{m \times d}$ and $a \in \mathbb{R}^m$. *Either* there is an $x \in \mathbb{R}^d$ such that $Ax = a$ and $x \geq 0$, *or* there is a vector $c \in \mathbb{R}^m$ such that $c^t A \geq 0$ and $c^t a < 0$, but not both.

Show that this version of Farkas' Lemma is equivalent to the one stated in the lecture (Proposition 2.61 of the lecture notes).
(Note: This is a corrected version – the vector c need not satisfy $c \geq 0$.)