Berlin Mathematical School

Freie Universität Berlin

Prof. Günter M. Ziegler
Albert Haase, Marie Litz

Institut für Mathematik
Arbeitsgruppe Diskrete Geometrie

Discrete Geometry 1 – Problem Sheet 6

Please hand in your solutions to Prof. Ziegler on **Wednesday, Nov. 27, 2013** before the lecture begins. Please put your name and student ID (if you have one) on the first page of your solutions and staple the sheets together.

Problem 1: *Unimodality of an f-Vector* (6 Points)

Let C_d be the d-dimensional cube, for instance $C_d = [0,1]^d$. For which $k = k(d)$ does C_d have the largest number of k-faces? To answer this, analyze the quotients f_k/f_{k-1}, and show that they decrease with k. Conclude that the f-vector of C_d is *unimodal*, that is,

$$f_0 < f_1 < \cdots < f_{k(d)} \geq f_{k(d)+1} > \cdots > f_{d-1}.$$

Problem 2: *The Half-Cube* (8 Points)

Let

$$H_d := \operatorname{conv}\{x \in \{0,1\}^d : x_1 + \cdots + x_d \in 2\mathbb{Z}\}$$

be the d-th *half-cube*.

(a) Describe the facets of H_d: How many are there and what are their combinatorial types? (The cases $d = 1, 2, 3$ need to be described separately.)

(b) Give an $\mathcal{H}$-representation of H_d. (The cases $d = 1, 2, 3$ need to be argued separately.)

(c) Show that H_d is *3-simplicial*, that is, all of its 3-faces are tetrahedra.

(d) Show that H_d is simplicial for $d \leq 4$, but not for $d > 4$.

Problem 3: *The f-Polynomial* (6 Points)

Define the *f-polynomial* of a d-polytope P with f-vector $f(P) = (1, f_0, \ldots, f_{d-1}, 1)$ as

$$f_P(t) := 1 + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d + t^{d+1}.$$

(a) Describe the f-polynomial $f_{P \times I}$ of the prism $P \times I$ in terms of the f-polynomial of P. Deduce from this a formula for the f-polynomial of the d-cube.

(b) Describe the f-polynomial of P^* in terms of the f-polynomial of P. Deduce from this a formula for the f-polynomial of the d-crosspolytope.

(c) Describe the f-polynomial of $P \times Q$ in terms of the f-polynomials of P and of Q.