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Discrete Geometry 1 – Problem Sheet 7

Please hand in your solutions to Prof. Ziegler on **Wednesday, Dec. 4, 2013** before the lecture begins. This is the last sheet of the first "half" of the semester.

Problem 1: *Joins, Constructing Polytopes* ((2+)4 Points)

Note: Problem 1 (a) has been corrected and turned into a bonus problem on Mon, Dec. 2. We will discuss it in the tutorials on Wednesday.

(a) *Bonus:* Let P and Q be polytopes each with the origin in its relative interior. For the purpose of this exercise we use the following definition of the join of P and Q:

$$P * Q = \operatorname{conv}\left\{ \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ y \\ -1 \end{pmatrix} \;:\; x \in P \text{ and } y \in Q \right\}.$$

Show that the join construction is self-dual in the following "strong" sense: The polar of the join $(P * Q)^*$ is affinely equivalent to the join of the polars $(P^* * Q^*)$.

(b) What is the smallest example of a polytope that is not combinatorially equivalent to a non-trivial join, product or direct sum of two polytopes? (Non-trivial means that we exclude cases like the join with the empty set, the product with a point etc.) Before you answer this, state how you interpret "smallest".

Problem 2: *What can an f-vector tell us?* (7(+2) Points)

Assume for the moment that

$$f = f(P) = (1, 15, 34, 28, 9, 1)$$

is the f-vector of a polytope P.

(a) What is the dimension of P?

(b) Is P simple or simplicial?

(c) Could P be a prism?

(d) Could P be a pyramid?

(e) Could P be a join?

(f) Could P be a stacked polytope $\mathrm{Stack}_d(d+1+n)$?

(g) *Bonus:* Is f the f-vector of a polytope?

Problem 3: *Stacked Polytopes and Trees* (7 Points)

Let $d \geq 3$ and $n \geq 3$. To every stacked polytope $\mathrm{Stack}_d(d+1+n)$ we can associate a graph-theoretic tree[1] $T = (V, E)$ by adding a vertex $v \in V$ for the simplex Δ_d and for every simplex we stack onto a facet. We then add an edge $e \in E$ for every two vertices in V which correspond to adjacent simplices. Two simplices are adjacent if one was stacked on a facet of the other.

(a) Do different trees correspond to different (combinatorial) types of polytopes?

(b) Do different types of stacked polytopes $\mathrm{Stack}_d(d+1+n)$ have different trees?

(c) Use (i) and (ii) to estimate the number of types of polytopes $\mathrm{Stack}_d(d+1+n)$ for fixed $d \geq 3$ and increasing n.

[1] For the definition of a "tree" see the excellent Wikipedia article http://en.wikipedia.org/wiki/Tree_(graph_theory).