Berlin Mathematical School

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Discrete Geometry 1 – Problem Sheet 8

Please hand in your solutions to Prof. Ziegler on **Wednesday, Dec. 11, 2013** before the lecture begins.

Problem 1: *k-Neighborliness* (6 Points)

Let P be a k-neighborly polytope of dimension $d \geq 1$. Show that every $(2k-1)$-face of P is a simplex. Conclude that if P is $(\lfloor \frac{d}{2} \rfloor + 1)$-neighborly, then P is a simplex.

Problem 2: *Cyclic Polytopes* (4(+2)+4 Points)

Consider the cyclic polytope

$$C_d(t_1, \dots, t_n) := \operatorname{conv}\{\gamma(t_1), \dots, \gamma(t_n)\} \subset \mathbb{R}^d,$$

where $t_1 < \cdots < t_n$ for $n > d > 1$ and γ denotes the moment curve in $\mathbb{R}^d$. In this problem we are referring to combinatorial types rather than geometric realizations.

(a) Let d be even. Let $\pi = (1\,2\,\dots n)$ be the cyclic permutation that sends $i \longmapsto i+1$ for $1 \leq i \leq n-1$ and $n \longmapsto 1$. Show that whenever $I \subset \{1, \dots, n\}$ is the index set (of the vertices) of a face of $C_d(t_1, \dots, t_n)$ the set $\pi(I)$ is an index set of a face of $C_d(t_1, \dots, t_n)$. Prove the same result for the permutation π' that sends $i \longmapsto n+1-i$, in other words, for

$$\pi' = \prod_{\substack{i<j \\ i+j=n+1}} (i, j).$$

(b) *Bonus:* Do the statements in (a) hold for odd d?

(c) Show that $C_d(d+1)$ is a d-simplex Δ_d and that $C_d(d+2)$ is the direct sum of simplices $\Delta_{\lceil d/2 \rceil} \oplus \Delta_{\lfloor d/2 \rfloor}$.

Problem 3: *Examples* (3+3 Points)

(a) Give an example of a (geometric) polytope P of dimension 4 with 1000 facets such that every two facets share a 2-face. By "share a 2-face" we mean that P has a 2-face incident to both facets.

(b) Give an example of a simplicial polytope of dimension 5 with 7 vertices whose combinatorial type is not that of a cyclic polytope $C_5(7)$.