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Discrete Geometry 1 – Problem Sheet 9

Please hand in your solutions to Prof. Ziegler on **Wednesday, Dec. 18, 2013** before the lecture begins.

Problem 1: *The "cube slice polytopes"* (17(+4) Points)

For $1 \le k \le d$ let

$$\begin{aligned}\Delta_{d-1}(k) &:= \{x \in [0,1]^d : x_1 + \cdots + x_d = k\} \\ &= \operatorname{conv}\{x \in \{0,1\}^d : x_1 + \cdots + x_d = k\}.\end{aligned}$$

(i) Show that $\Delta_{d-1}(k)$ is affinely equivalent to

$$\begin{aligned}\Delta'_{d-1}(k) &:= \{x \in [0,1]^{d-1} : k-1 \le x_1 + \cdots + x_{d-1} \le k\} \\ &= \operatorname{conv}\{x \in \{0,1\}^{d-1} : k-1 \le x_1 + \cdots + x_{d-1} \le k\}.\end{aligned}$$

(2 Points)

(ii) Describe $\Delta_3(2)$. (1 Point)

(iii) Study how the hyperplane $H_k = \{x \in \mathbb{R}^d : x_1 + \cdots + x_d = k\}$ cuts the faces of the d-cube $[0,1]^d$. What do the resulting faces look like? Conversely, describe how the faces of $\Delta_{d-1}(k)$ arise from faces of $[0,1]^d$. (Hint: Distinguish vertices from higher dimensional faces!) (2 Points)

(iv) Let $k \in \{1, \ldots, d-1\}$. Show that the $\mathcal{H}$-description and the $\mathcal{V}$-description in the definition of $\Delta_{d-1}(k)$ give the same $(d-1)$-polytope. (2 Points)

(v) Show that $\Delta_{d-1}(k)$ and $\Delta_{d-1}(d-k)$ are combinatorially equivalent. (1 Point)

(vi) Show that for even d, $\Delta_{d-1}(\frac{d}{2})$ is *centrally symmetric*, i.e. there is a center point c such that for all $x \in \mathbb{R}^d$, $c+x \in \Delta_{d-1}(\frac{d}{2})$ if and only if $c-x \in \Delta_{d-1}(\frac{d}{2})$. (1 Point)

(vii) Show that $\Delta_{d-1}(1)$ and $\Delta_{d-1}(d-1)$ are simplices. (1 Point)

(viii) Show that for $1 < k < d-1$, $\Delta_{d-1}(k)$ has $2d$ facets. What are their combinatorial types? There are two different combinatorial types, except in the case

$k = \frac{d}{2}$. (2 Points)

(ix) A polyope is *k-simple* if its dual is k-simplicial. Show that $\Delta_{d-1}(k)$ is 2-simplicial and $(d-2)$-simple. (2 Points)

(x) Describe $\Delta_4(2)$: compute the f-vector, describe the facets. (1 Point)

(xi) *Bonus:* Show that the f-vector of $\Delta_{d-1}(k)$ is given by

$$\begin{aligned} f_{i-1}(\Delta_{d-1}(k)) &= \big|\{[d] = A \uplus B \uplus C \,:\, |A| < k, |B| < d-k, |C| = i\}\big| \\ &= \sum_{\substack{0 \le s < k \\ k < s+i \le d}} \binom{d}{s}\binom{d-s}{i} \\ &= \sum_{\max\{-1,k-i\} < s < \min\{k,d-i+1\}} \frac{d!}{s!i!(d-s-i)!} \end{aligned}$$

for $i > 1$. How about f_0?

(Hint: Every i-face of $[0,1]^d$ can be described in the form

$$\{x \in \mathbb{R}^d : x_j = 1 \text{ for } j \in A,\ x_j = 0 \text{ for } j \in B,\ 0 \le x_j \le 1 \text{ for } j \in C\},$$

for suitable sets A, B, C satisfying $A \uplus B \uplus C = [d]$ and $|C| = i$.) ((4) Points)

(xii) Compute and plot the f-vector of $\Delta_{41}(21)$. You may use (xi). (2 Points)