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Discrete Geometry 1 – Xmas Problem Sheet

These are bonus problems for the Xmas break – have fun with them.
If you want them graded or would like credit, please hand in your solutions to Dr. Izmestiev on **Wednesday, Jan. 8, 2014** before the lecture begins. (We'll count the bonus points for the first or second half of the semester, as you like. Please specify.)

Problem 1: *Triangles or simple vertices* ((4) Points)

Show that every 3-dimensional polytope has at least 4 triangle faces or at least 4 simple vertices (i.e., vertices of degree 3) or both. Show that 3-polytopes with only 4 triangle faces and exactly 4 simple vertices but an arbitrarily large number of edges exist.

Problem 2: *f-vectors of 3-polytopes* ((6)+(6) Points)

(i) Show that the f-vectors of 3-polytopes are given by the set

$$\{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 = 2, \quad f_2 \le 2f_0 - 4, \quad f_0 \le 2f_2 - 4\}.$$

(ii) Show that the f-vectors of centrally-symmetric 3-polytopes are given by

$$\{(f_0, f_1, f_2) \in (2\mathbb{Z})^3 : f_0 - f_1 + f_2 = 2, \quad f_2 \le 2f_0 - 4, \quad f_0 \le 2f_2 - 4, \quad f_0 + f_2 \ge 14\}.$$

Problem 3: *The "fractional cube slice polytopes"*

For an odd integer ℓ, $1 \le \ell \le 2d - 1$ let

$$\Delta_{d-1}(\tfrac{\ell}{2}) := \{x \in [0,1]^d : x_1 + \cdots + x_d = \tfrac{\ell}{2}\}.$$

(i) Show that $\Delta_{d-1}(k)$ is affinely equivalent to ((2) Points)

$$\Delta'_{d-1}(\tfrac{\ell}{2}) := \{x \in [0,1]^{d-1} : \tfrac{\ell}{2} - 1 \le x_1 + \cdots + x_{d-1} \le \tfrac{\ell}{2}\}.$$

(ii) Describe $\Delta_2(\frac{3}{2})$. ((1) Point)

(iii) Describe $\Delta_3(\frac{3}{2})$. (It is a *semi-regular* polytope: Its faces are regular polygons, and at each vertex it has the same number and types of faces.) ((1) Point)

(iv) Study how the hyperplane $H_{\ell/2} = \{x \in \mathbb{R}^d : x_1 + \cdots + x_d = \frac{\ell}{2}\}$ cuts the faces of the d-cube $[0,1]^d$. What do the resulting faces look like? Conversely, describe how the faces of $\Delta_{d-1}(\frac{\ell}{2})$ arise from faces of $[0,1]^d$. ((2) Points)

(v) Give a $\mathcal{V}$-description of $\Delta_{d-1}(\frac{\ell}{2})$. ((1) Point)

(vi) Show that $\Delta_{d-1}(\frac{\ell}{2})$ and $\Delta_{d-1}(d - \frac{\ell}{2})$ are congruent. Derive that for odd d, $\Delta_{d-1}(\frac{d}{2})$ is centrally symmetric. ((1) Point)

(vii) For $\Delta_{d-1}(\frac{1}{2})$ and $\Delta_{d-1}(\frac{2d-1}{2})$ are simplices. Show that for $3 \leq \ell \leq 2d-3$, $\Delta_{d-1}(\frac{\ell}{2})$ has $2d$ facets. What are their combinatorial types? There are two different combinatorial types, except in the case $\ell = d$, for odd d. ((2) Points)

(viii) State and prove a formula for the f-vector of $\Delta_{d-1}(\frac{\ell}{2})$. ((4) Points)

(ix) Compute and plot the f-vectors of $\Delta_{42}(\frac{13}{2})$, $\Delta_{42}(\frac{23}{2})$, $\Delta_{42}(\frac{33}{2})$, and $\Delta_{42}(\frac{43}{2})$. ((2) Points)