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Discrete Geometry 1

Solutions for the remaining exercise of the Xmas Sheet

Problem 3: *The "fractional cube slice polytopes"*
For an odd integer ℓ, $1 \leq \ell \leq 2d-1$ let

$$\Delta_{d-1}(\tfrac{\ell}{2}) := \{x \in [0,1]^d : x_1 + \cdots + x_d = \tfrac{\ell}{2}\}.$$

(iv) Study how the hyperplane $H_{\ell/2} = \{x \in \mathbb{R}^d : x_1 + \cdots + x_d = \frac{\ell}{2}\}$ cuts the faces of the d-cube $[0,1]^d$. What do the resulting faces look like? Conversely, describe how the faces of $\Delta_{d-1}(\frac{\ell}{2})$ arise from faces of $[0,1]^d$. ((2) Points)

The reasoning goes as in the previous problem sheet. Every k-face of the d-cube is described by the set of coordinate entries which are fixed to 1, which we call A, the set of coordinate entries which are fixed to 0, which we call B, and the set of entries that varies between 0 and 1, called C, and $|C| = k$. Denote $|A|$ with a. Then the resulting face of the fractional hypersimplex is combinatorially equivalent to $\Delta_{k-1}(\frac{\ell}{2} - a)$.
Vice versa every $(k-1)$-face of $\Delta_{d-1}(\frac{\ell}{2})$ gives rise to a k-dimensional face of the cube. This face is defined by the fixed 0 and 1 entries of $\Delta_{d-1}(\frac{\ell}{2})$.

(v) Give a $\mathcal{V}$-description of $\Delta_{d-1}(\frac{\ell}{2})$. ((1) Point)

The vertices of $\Delta_{d-1}(\frac{\ell}{2})$ are the midpoints certain edges, namely the edges which connect one vertex of the cube whose entries sum to $\frac{\ell-1}{2}$ and one whose entries sum to $\frac{\ell+1}{2}$.

$$\mathrm{vert}(\Delta_{d-1}(\tfrac{\ell}{2})) = \{x \in \mathbb{R}^d : x \text{ has } \tfrac{\ell-1}{2} \text{ ones, one } \tfrac{1}{2} \text{ and zeros otherwise}\}$$

(vi) Show that $\Delta_{d-1}(\frac{\ell}{2})$ and $\Delta_{d-1}(d - \frac{\ell}{2})$ are congruent. Derive that for odd d, $\Delta_{d-1}(\frac{d}{2})$ is centrally symmetric. ((1) Point)

Just as in the last exercise sheet, the map $x \longmapsto \mathbb{1} - x$ maps $\Delta_{d-1}(\frac{\ell}{2})$ to $\Delta_{d-1}(d - \frac{\ell}{2})$. The map is a reflection through the centroid $(\frac{1}{2}, \dots, \frac{1}{2})$. Therefore the polytopes are congruent. We have $d - \frac{d}{2} = \frac{d}{2}$, so for odd d we can build the fractional hypersimplex $\Delta_{d-1}(\frac{d}{2}) = \Delta_{d-1}(d - \frac{d}{2}) = -\Delta_{d-1}(\frac{d}{2})$.

(vii) For $\Delta_{d-1}(\frac{1}{2})$ and $\Delta_{d-1}(\frac{2d-1}{2})$ are simplices. Show that for $3 \leq \ell \leq 2d-3$, $\Delta_{d-1}(\frac{\ell}{2})$ has $2d$ facets. What are their combinatorial types? There are two different combinatorial types, except in the case $\ell = d$, for odd d. ((2) Points)

The cube is simple, so $\Delta_{d-1}(\frac{1}{2})$ and $\Delta_{d-1}(\frac{2d-1}{2})$ must be simplices. For all other possible polytopes $\Delta_{d-1}(\frac{\ell}{2})$, we look at the facets of the cube. Each facet is characterized by a coordinate i which is fixed to 1 or 0. For all $3 \leq \ell \leq 2d-3$, there exist $x_j \in (0,1)$, $j \in [d] \setminus \{i\}$, with one value x_i set to 1 or 0 that satisfy $x_1 + \cdots + x_d = \frac{\ell}{2}$, so every facet as an interior point that lies in $\Delta_{d-1}(\frac{\ell}{2})$. Thus every facet of the cube yields at least one face of the hypersimplex. It cannot be more than one because we intersected with a hyperplane and $\Delta_{d-1}(\frac{\ell}{2})$ does not contain a whole face of the cube.

(viii) State and prove a formula for the f-vector of $\Delta_{d-1}(\frac{\ell}{2})$. ((4) Points)

Again, following the line of reasoning from the previous exercise sheet, we obtain for all $1 \leq i \leq d$

$$\begin{aligned} f_{i-1}(\Delta_{d-1}(\tfrac{\ell}{2})) &= \left|\{[d] = A \uplus B \uplus C \;:\; |A| < \tfrac{\ell}{2}, |B| < d - \tfrac{\ell}{2}, |C| = i\}\right| \\ &= \sum_{\substack{0 \leq s \leq \frac{\ell-1}{2} \\ \frac{\ell+1}{2} \leq s+i \leq d}} \binom{d}{s}\binom{d-s}{i} \\ &= \sum_{\max\{0, \frac{\ell+1}{2} - i\} \leq s \leq \min\{\frac{\ell-1}{2}, d-i\}} \frac{d!}{s!i!(d-s-i)!}. \end{aligned}$$

(ix) Compute and plot the f-vectors of $\Delta_{42}(\frac{13}{2})$, $\Delta_{42}(\frac{23}{2})$, $\Delta_{42}(\frac{33}{2})$, and $\Delta_{42}(\frac{43}{2})$. ((2) Points)

The f-vectors were computed in sage (which is accessible on all math/computer science computers at the FU and can be downloaded (`http://www.sagemath.org/`) and used online (`http://www.sagenb.org/` or `https://cloud.sagemath.com/`).

The necessary syntax:

- `range (k,l)` gives $[k, k+1, \ldots, l-1]$ and `range (l)` gives $[0, \ldots, l-1]$,
- `sum f(x) for x in range (k,l)` gives the sum
- For given n `[binomial(n,m) for m in range (k,l)]` computes $[\binom{n}{k}, \ldots, \binom{n}{l-1}]$
- `zip(a,b)` creates a list of tuples (a_i, b_i) from lists a,b
- `list_plot(c)` plots a list c of tuples

Now, for the first polytope, we obtain

```
sage:   d=43
sage:   l=13
sage:   f=[sum(binomial(d,s)*binomial(d-s,i) for s in range (max(0,
        (l+1)/2-i),1+min((l-1)/2,d-i))) for i in range (1,d)]
[225568798,
4736944758,
...
3612,
86]
sage:   list_plot(zip(range(1,d),f))
```

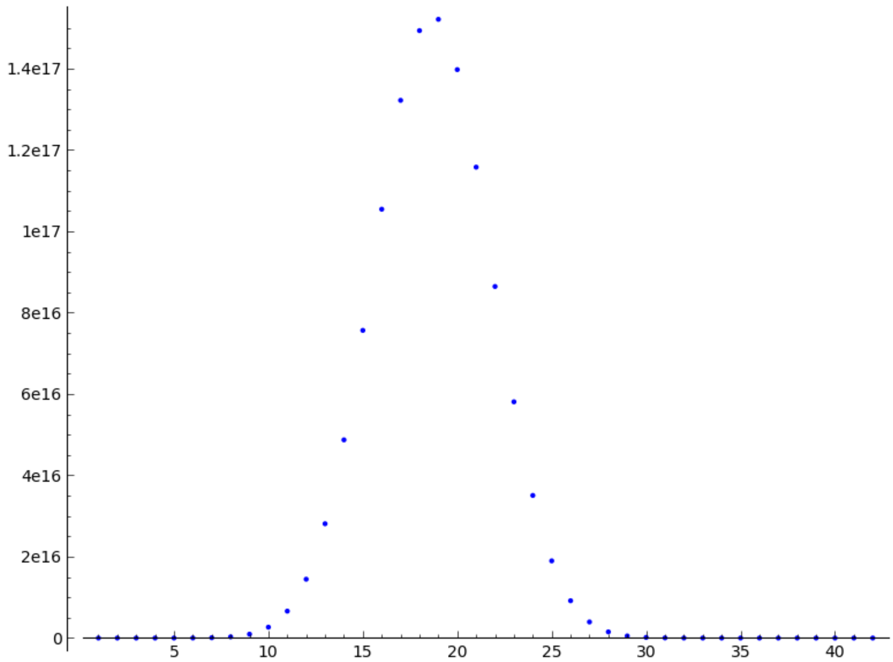

Figure 1: f-vector of $\Delta_{42}(\frac{13}{2})$

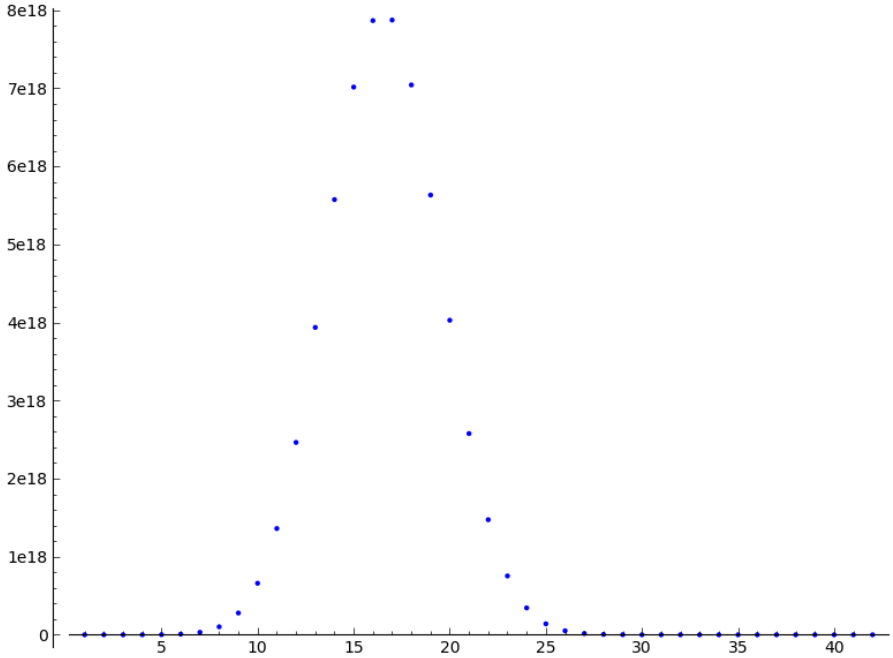

Figure 2: f-vector of $\Delta_{42}(\frac{23}{2})$

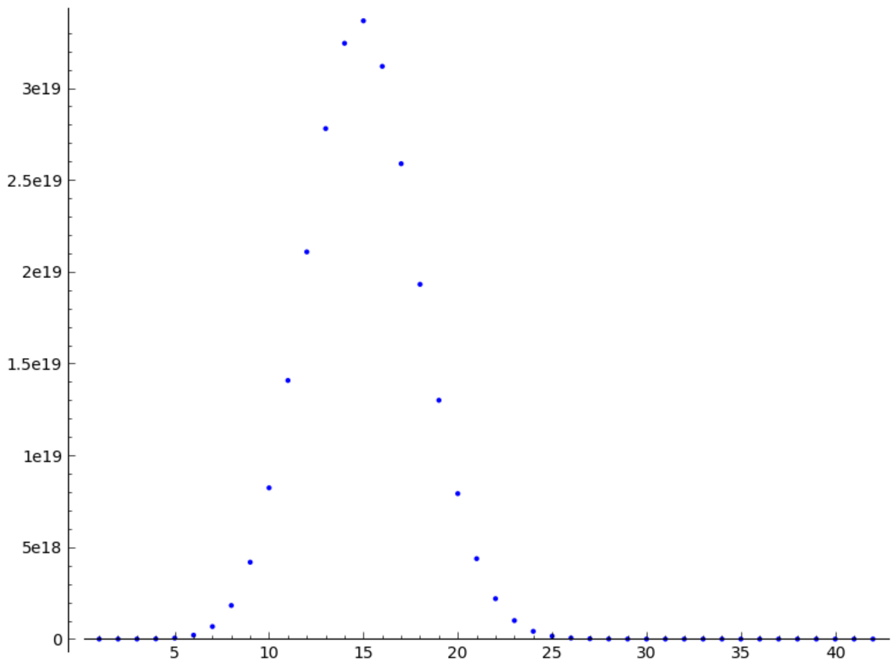

Figure 3: f-vector of $\Delta_{42}(\frac{33}{2})$

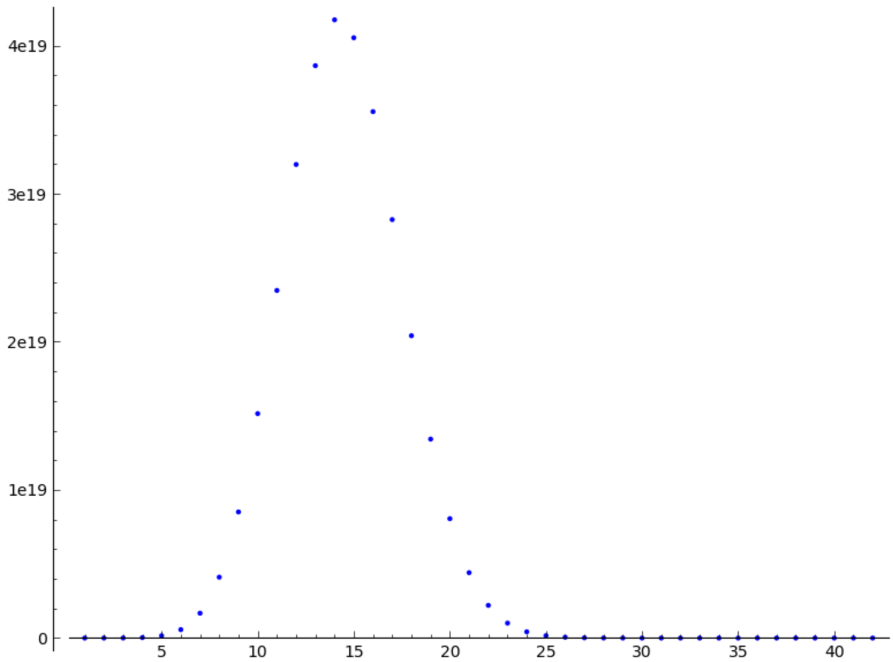

Figure 4: f-vector of $\Delta_{42}(\frac{43}{2})$

Apparently, for increasing ℓ the maximum of the curve shifts to the left.