

# Discrete Geometry I

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— Preliminary Lecture Notes (without any guarantees) —

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This is the first in a series of three courses on Discrete Geometry. We will get to know fascinating geometric structures such as configurations of points and lines, hyperplane arrangements, and in particular polytopes and polyhedra, and learn how to handle them using modern methods for computation and visualization and current analysis and proof techniques. A lot of this looks quite simple and concrete at first sight (and some of it is), but it also very quickly touches topics of current research.

For students with an interest in discrete mathematics and geometry, this is the starting point to specialize in discrete geometry. The topics addressed in the course supplement and deepen the understanding of discrete-geometric structures appearing in differential geometry, optimization, combinatorics, topology, and algebraic geometry. To follow the course, a solid background in linear algebra is necessary. Some knowledge of combinatorics and geometry is helpful.

## Basic Literature

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A rough schedule, which we will adapt as we move along:

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3. ....	22. October
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8. ....	[?] 5. November
9. ....	[?] 6. November
10. ....	12. November
11. ....	13. November
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14. ....	26. November
15. ....	27. November
16. ....	[?] 3. December
17. ....	4. December
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21. ....	18. December
22. ....	[?] 7. January
23. ....	[?] 8. January
24. ....	14. January
25. ....	15. January
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27. ....	22. January
28. ....	28. January
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## 0 Introduction

### What's the goal?

This is a first course in a large and interesting mathematical domain commonly known as “Discrete Geometry”. This spans from very classical topics (such as regular polyhedra – see Euclid’s *Elements*) to very current research topics (Discrete Geometry, Extremal Geometry, Computational Geometry, Convex Geometry) that are also of great industrial importance (for Computer Graphics, Visualization, Molecular Modelling, and many other topics).

My goal will be to develop these topics in a three-semester sequence of Graduate Courses in such a way that

- you get an **overview** of the field of Discrete Geometry and its manifold connections,
- you learn to understand, analyze, visualize, and confidently/competently argue about the basic **structures** of Discrete Geometry, which includes
  - point configurations/hyperplane arrangements,
  - frameworks
  - subspace arrangements, and
  - polytopes and polyhedra,
- you learn to know (and appreciate) the most important **results** in Discrete Geometry, which includes both simple & basic as well as striking key results,
- you get to learn and practice important **ideas and techniques** from Discrete Geometry (many of which are interesting also for other domains of Mathematics), and
- You learn about current **research topics** and problems treated in Discrete Geometry.

# 1 Some highlights to start with

## 1.1 Point configurations

**Proposition 1.1** (Sylvester–Gallai 1893/1944). *Every finite set of  $n$  points in the plane, not all on a line,  $n$  large, defines an “ordinary” line, which contain exactly 2 of the points.*

The “BOOK proof” for this result is due to L. M. Kelly [1].

**Theorem/Problem 1.2** (Green–Tao 2012 [4]). *Every finite set of  $n$  points in the plane, not all on a line,  $n$  large, defines at least  $n/2$  “ordinary” lines, which contain exactly 2 of the points. How large does  $n$  have to be for this to be true?  $n > 13$ ?*

**Theorem/Problem 1.3** (Blagojevic–Matschke–Ziegler 2009 [2]). *For  $d \geq 1$  and a prime  $r$ , any  $(r - 1)(d + 1) + 1$  colored points in  $\mathbb{R}^d$ , where no  $r$  points have the same color, can be partitioned into  $r$  “rainbow” subsets, in which no 2 points have the same color, such that the convex hulls of the  $r$  blocks have a point in common.*

*Is this also true if  $r$  is not a prime? How about  $d = 2$  and  $r = 4$ , cf. [6]?*

## 1.2 Polytopes

**Theorem 1.4** (Schläfli 1852). *The complete classification of regular polytopes in  $\mathbb{R}^d$ :*

- $d$ -simplex ( $d \geq 1$ )
- the regular  $n$ -gon ( $d = 2, n \geq 3$ )
- $d$ -cube and  $d$ -crosspolytope ( $d \geq 2$ )
- icosahedron and dodecahedron ( $d = 3$ )
- 24-cell ( $d = 4$ )
- 120-cell and 600-cell ( $d = 4$ )

**Theorem/Problem 1.5** (Santos 2012 [9]). *There is a simple polytope of dimension  $d = 43$  and  $n = 86$  facets, whose graph diameter is not, as conjectured by Hirsch (1957), at most 43.*

*What is the largest possible graph diameter for a  $d$ -dimensional polytope with  $n$  facets? Is it a polynomial function of  $n$ ?*

## 1.3 Sphere configurations/packings/tilings

**Theorem/Problem 1.6** (see [8]). *For  $d \geq 2$ , the kissing number  $\kappa_d$  denotes the maximal number of non-overlapping unit spheres that can simultaneously touch (“kiss”) a given unit sphere in  $\mathbb{R}^d$ .*

$d = 2$ :  $\kappa_2 = 6$ , “hexagon configuration”, unique

$d = 3$ :  $\kappa_3 = 12$ , “dodecahedron configuration”, not unique

$d = 4$ :  $\kappa_4 = 24$  (Musin 2008 [7]) “24-cell”, unique?

$d = 8$ :  $\kappa_8 = 240$ ,  $E_8$  lattice, unique?

$d = 24$ :  $\kappa_{24} = 196560$ , “Leech lattice”, unique?

**Theorem/Problem 1.7** (Engel 1980 [3] [5] [10]). *There is a stereohedron (that is, a 3-dimensional polytope whose congruent copies tile  $\mathbb{R}^3$ ) with 38 facets. But is the maximal number of facets of a stereohedron in  $\mathbb{R}^3$  bounded at all?*

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## 2 Basic structures in discrete geometry

### 2.1 Convex sets, intersections and separation

#### 2.1.1 Convex sets

Geometry in  $\mathbb{R}^d$  (or in any finite-dimensional vector space over a real closed field ...)

**Definition 2.1** (Convex set). A set  $S \subseteq \mathbb{R}^d$  is *convex* if  $\lambda p + \mu q \in S$  for all  $p, q \in S$ ,  $\lambda, \mu \in \mathbb{R}_{\geq 0}$ ,  $\lambda + \mu = 1$ .

**Lemma 2.2.**  $S \subseteq \mathbb{R}^d$  is convex if and only if  $\sum_{i=1}^k \lambda_i x_i \in S$  for all  $k \geq 1$ ,  $x_1, \dots, x_k \in S$ ,  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ ,  $\lambda_1, \dots, \lambda_k \geq 0$ ,  $\sum_{i=1}^k \lambda_i = 1$ .

*Proof.* For “if” take the special case  $k = 2$ .

For “only if” we use induction on  $k$ , where the case  $k = 1$  is vacuous and  $k = 2$  is clear. Without loss of generality,  $0 < \lambda_k < 1$ . Now rewrite  $\sum_{i=1}^k \lambda_i x_i$  as

$$(1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i + \lambda_k x_k$$

□

Compare:

- $U \subseteq \mathbb{R}^d$  is a *linear subspace* if  $\lambda p + \mu q \in U$  for all  $p, q \in U$ ,  $\lambda, \mu \in \mathbb{R}$ .
- $U \subseteq \mathbb{R}^d$  is an *affine subspace* if  $\lambda p + \mu q \in U$  for all  $p, q \in U$ ,  $\lambda, \mu \in \mathbb{R}$ ,  $\lambda + \mu = 1$ .

#### 2.1.2 Operations on convex sets

**Lemma 2.3** (Operations on convex sets). Let  $K, K' \subseteq \mathbb{R}^d$  be convex sets.

- $K \cap K' \subseteq \mathbb{R}^d$  is convex.
- $K \times K' \subseteq \mathbb{R}^{d+d}$  is convex.
- For any affine map  $f : \mathbb{R}^d \rightarrow \mathbb{R}^e$ ,  $x \mapsto Ax + b$ , the image  $f(K)$  is convex.
- The Minkowski sum  $K + K' := \{x + y : x \in K, y \in K'\}$  is convex.

**Exercise 2.4.** Interpret the Minkowski sum as the image of an affine map applied to a product.

**Lemma 2.5.** Hyperplanes  $H = \{x \in \mathbb{R}^d : a^t x = \beta\}$  are convex.

Open halfspaces  $H^+ = \{x \in \mathbb{R}^d : a^t x > \beta\}$  and  $H^- = \{x \in \mathbb{R}^d : a^t x < \beta\}$  are convex.

Closed halfspaces  $\overline{H}^+ = \{x \in \mathbb{R}^d : a^t x \geq \beta\}$  and  $\overline{H}^- = \{x \in \mathbb{R}^d : a^t x \leq \beta\}$  are convex.

More generally, for  $A \in \mathbb{R}^{n \times d}$  and  $b \in \mathbb{R}^n$ ,

- $\{x \in \mathbb{R}^d : Ax = 0\}$  is a linear subspace,
- $\{x \in \mathbb{R}^d : Ax = b\}$  is an affine subspace,
- $\{x \in \mathbb{R}^d : Ax \leq b\}$  and  $\{x \in \mathbb{R}^d : Ax \leq b\}$  are convex subsets of  $\mathbb{R}^d$ .



### 2.1.3 Convex hulls, Radon's lemma and Helly's theorem

**Definition 2.6** (convex hull). For any  $S \subseteq \mathbb{R}^d$ , the *convex hull* of  $S$  is defined as

$$\text{conv}(S) := \bigcap \{K \subseteq \mathbb{R}^d : K \text{ convex}, S \subseteq K \subseteq \mathbb{R}^d\}.$$

Note the analogy to the usual definition of *affine hull* (an affine subspace) and *linear hull* (or *span*), a vector subspace.

**Exercise 2.7.** Show that

- $\text{conv}(S)$  is convex,
- $S \subseteq \text{conv}(S)$ ,
- $S \subseteq S'$  implies  $\text{conv}(S) \subseteq \text{conv}(S')$ ,
- $\text{conv}(S) = S$  if  $S$  is convex, and
- $\text{conv}(\text{conv}(S)) = \text{conv}(S)$ .

**Lemma 2.8** (Radon's<sup>1</sup> lemma). Any  $d + 2$  points  $p_1, \dots, p_{d+2} \in \mathbb{R}^d$  can be partitioned into two groups  $(p_i)_{i \in I}$  and  $(p_i)_{i \notin I}$  whose convex hulls intersect.

*Proof.* The  $d + 2$  vectors  $\begin{pmatrix} p_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} p_{d+2} \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}$  are linearly dependent,

$$\lambda_1 \begin{pmatrix} p_1 \\ 1 \end{pmatrix} + \dots + \lambda_{d+2} \begin{pmatrix} p_{d+2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here not all  $\lambda_i$ 's are zero, so some are positive, some are negative, and we can take  $I := \{i : \lambda_i > 0\} \neq \emptyset$ . Thus with  $\Lambda := \sum_{i \in I} \lambda_i > 0$  we can rewrite the above equation as

$$\sum_{i \in I} \frac{\lambda_i}{\Lambda} p_i = \sum_{i \notin I} \frac{-\lambda_i}{\Lambda} p_i.$$

□

Note that even more so Radon's lemma holds for any  $n \geq d + 2$  points in  $\mathbb{R}^d$ .

**Theorem 2.9** (Helly's Theorem). Let  $C_1, \dots, C_N$  be a finite family of  $N \geq d + 1$  convex sets such that any  $d + 1$  of them have a non-empty intersection. Then the intersection of all  $N$  of them is non-empty as well.

*Proof.* This is trivial for  $N = d + 1$ . Assume  $N \geq d + 2$ . We use induction on  $N$ .

By induction, for each  $i$  there is a point  $\bar{p}_i$  that lies in all  $C_j$  except for possibly  $C_i$ . Now form a Radon partition of the points  $\bar{p}_i$ , and let  $p$  be a corresponding intersection point. About this point we find that on the one hand it lies in all  $C_i$  except for possibly those with  $i \in I$ , and on the other hand it lies in all  $C_i$  except for possibly those with  $i \notin I$ . □

Note that the claim of Helly's theorem does not follow if we only require that any  $d$  sets intersect (take the  $C_i$  to be hyperplanes in general position!) or if we admit infinitely many convex sets (take  $C_i := [i, \infty)$ ).

End of class on October 16

<sup>1</sup>In class, I called this Carathéodory's lemma, which was wrong – Carathéodory's lemma is a related result, which you will see on the problem set.