Discrete Geometry I

`http://wikis.fu-berlin.de/display/discgeom/Discrete+Geometry+I`

— Preliminary Lecture Notes (without any guarantees) —

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This is the first in a series of three courses on Discrete Geometry. We will get to know fascinating geometric structures such as configurations of points and lines, hyperplane arrangements, and in particular polytopes and polyhedra, and learn how to handle them using modern methods for computation and visualization and current analysis and proof techniques. A lot of this looks quite simple and concrete at first sight (and some of it is), but it also very quickly touches topics of current research.

For students with an interest in discrete mathematics and geometry, this is the starting point to specialize in discrete geometry. The topics addressed in the course supplement and deepen the understanding of discrete-geometric structures appearing in differential geometry, optimization, combinatorics, topology, and algebraic geometry. To follow the course, a solid background in linear algebra is necessary. Some knowledge of combinatorics and geometry is helpful.

Basic Literature

[1] Satyan L. Devadoss and Joseph O'Rourke. *Discrete and Computational Geometry*. Princeton University Press, Princeton NJ, 2011.

[2] Jacob E. Goodman and Joseph O'Rourke, editors. *Handbook of Discrete and Computational Geometry*. Chapman & Hall/CRC Press, Boca Raton, second edition, 2004.

[3] Peter M. Gruber. *Convex and Discrete Geometry*, volume 336 of *Grundlehren Series*. Springer, 2007.

[4] Peter M. Gruber and Jörg Wills, editors. *Handbook of Convex Geometry*. North-Holland, Amsterdam, 1993. 2 Volumes.

[5] Branko Grünbaum. *Convex Polytopes*, volume 221 of *Graduate Texts in Math.* Springer-Verlag, New York, 2003. Second edition prepared by V. Kaibel, V. Klee and G. M. Ziegler (original edition: Interscience, London 1967).

[6] Michael Joswig and Thorsten Theobald. *Polyhedral and Algebraic Methods in Computational Geometry*. Universitext. Springer, 2013.

[7] Jiří Matoušek and Bernd Gärtner. *Understanding and Using Linear Programming*. Universitext. Springer, 2007.

[8] Günter M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Revised edition, 1998; seventh updated printing 2007.

Contents

A rough schedule, which we will adapt as we move along:

0   Introduction

What's the goal?

This is a first course in a large and interesting mathematical domain commonly known as "Discrete Geometry". This spans from very classical topics (such as regular polyhedra – see Euclid's *Elements*) to very current research topics (Discrete Geometry, Extremal Geometry, Computational Geometry, Convex Geometry) that are also of great industrial importance (for Computer Graphics, Visualization, Molecular Modelling, and many other topics).

My goal will be to develop these topics in a three-semester sequence of Graduate Courses in such a way that

- you get an **overview** of the field of Discrete Geometry and its manifold connections,

- you learn to understand, analyze, visualize, and confidently/competently argue about the basic **structures** of Discrete Geometry, which includes
    - point configurations/hyperplane arrangements,
    - frameworks
    - subspace arrangements, and
    - polytopes and polyhedra,

- you learn to know (and appreciate) the most important **results** in Discrete Geometry, which includes both simple & basic as well as striking key results,

- you get to learn and practice important **ideas and techniques** from Discrete Geometry (many of which are interesting also for other domains of Mathematics), and

- You learn about current **research topics** and problems treated in Discrete Geometry.

1 Some highlights to start with

1.1 Point configurations

Proposition 1.1 (Sylvester–Gallai 1893/1944). *Every finite set of n points in the plane, not all on a line, n large, defines an "ordinary" line, which contain exactly 2 of the points.*

The "BOOK proof" for this result is due to L. M. Kelly [1].

Theorem/Problem 1.2 (Green–Tao 2012 [4]). *Every finite set of n points in the plane, not all on a line, n large, defines at least $n/2$ "ordinary" lines, which contain exactly 2 of the points. How large does n have to be for this to be true? $n > 13$?*

Theorem/Problem 1.3 (Blagojevic–Matschke–Ziegler 2009 [2]). *For $d \geq 1$ and a prime r, any $(r-1)(d+1)+1$ colored points in $\mathbb{R}^d$, where no r points have the same color, can be partitioned into r "rainbow" subsets, in which no 2 points have the same color, such that the convex hulls of the r blocks have a point in common. Is this also true if r is not a prime? How about $d = 2$ and $r = 4$, cf. [6]?*

1.2 Polytopes

Theorem 1.4 (Schläfli 1852). *The complete classification of regular polytopes in $\mathbb{R}^d$:*
– *d-simplex ($d \geq 1$)*
– *the regular n-gon ($d = 2$, $n \geq 3$)*
– *d-cube and d-crosspolytope ($d \geq 2$)*
– *icosahedron and dodecahedron ($d = 3$)*
– *24-cell ($d = 4$)*
– *120-cell and 600-cell ($d = 4$)*

Theorem/Problem 1.5 (Santos 2012 [9]). *There is a simple polytope of dimension $d = 43$ and $n = 86$ facets, whose graph diameter is not, as conjectured by Hirsch (1957), at most 43. What is the largest possible graph diameter for a d-dimensional polytope with n facets? Is it a polynomial function of n?*

1.3 Sphere configurations/packings/tilings

Theorem/Problem 1.6 (see [8]). *For $d \geq 2$, the kissing number κ_d denotes the maximal number of non-overlapping unit spheres that can simultaneously touch ("kiss") a given unit sphere in $\mathbb{R}^d$.*
$d = 2$: $\kappa_2 = 6$, "hexagon configuration", unique
$d = 3$: $\kappa_3 = 12$, "dodecahedron configuration", not unique
$d = 4$: $\kappa_4 = 24$ (Musin 2008 [7]) "24-cell", unique?
$d = 8$: $\kappa_8 = 240$, E_8 lattice, unique?
$d = 24$: $\kappa_{24} = 196560$, "Leech lattice", unique?

Theorem/Problem 1.7 (Engel 1980 [3] [5] [10])**.** *There is a stereohedron (that is, a 3-dimensional polytope whose congruent copies tile $\mathbb{R}^3$) with 38 facets. But is the maximal number of facets of a stereohedron in $\mathbb{R}^3$ bounded at all?*

[1] Martin Aigner and Günter M. Ziegler. *Proofs from THE BOOK*. Springer-Verlag, Heidelberg Berlin, fourth edition, 2009.

[2] Pavle V. M. Blagojević, Benjamin Matschke, and Günter M. Ziegler. Optimal bounds for the colored Tverberg problem. Preprint, October 2009, 10 pages; revised November 2009, 11 pages; J. European Math. Soc., to appear; `http://arXiv.org/abs/0910.4987`.

[3] Peter Engel. Über Wirkungsbereichsteilungen von kubischer Symmetrie. *Zeitschrift f. Kristallographie*, 154:199–215, 1981.

[4] Ben Green and Terence Tao. On sets defining few ordinary lines. Preprint, August 2012, 72 pages, `http://arxiv.org/abs/1208.4714`.

[5] Branko Grünbaum and Geoffrey C. Shephard. Tilings with congruent tiles. *Bulletin Amer. Math. Soc.*, 3:951–973, 1980.

[6] Benjamin Matschke and Günter M. Ziegler. Die Rätselseite: Zehn bunte Punkte in der Ebene. *Mitteilungen der DMV*, 18(3):171, 2010. `http://page.math.tu-berlin.de/~mdmv/archive/18/mdmv-18-3-171.pdf`.

[7] Oleg R. Musin. The kissing number in four dimensions. *Annals of Mathematics*, 168:1–32, 2008.

[8] Florian Pfender and Günter M. Ziegler. Kissing numbers, sphere packings, and some unexpected proofs. *Notices of the AMS*, 51(8):873–883, September 2004.

[9] Francisco Santos. A counterexample to the Hirsch conjecture. *Annals of Math.*, 176:383–412, 2012.

[10] Moritz Schmitt and Günter M. Ziegler. Ten problems. In M. Senechal, editor, *Shaping Space. Exploring Polyhedra in Nature, Art, and the Geometrical Imagination*, pages 279–289 and 315–319. Springer, New York, 2013.

End of class on October 15

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2   Basic structures in discrete geometry

2.1   Convex sets, intersections and separation

2.1.1   Convex sets

Geometry in $\mathbb{R}^d$ (or in any finite-dimensional vector space over a real closed field ...)

Definition 2.1 (Convex set)**.** A set $S \subseteq \mathbb{R}^d$ is *convex* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}_{\geq 0}$, $\lambda + \mu = 1$.

Lemma 2.2. $S \subseteq \mathbb{R}^d$ *is convex if and only if* $\sum_{i=1}^{k} \lambda_i x_i \in S$ *for all* $k \geq 1$, $x_1, \ldots, x_k \in S$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, $\lambda_1, \ldots, \lambda_k \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$.

Proof. For "if" take the special case $k = 2$.

For "only if" we use induction on k, where the case $k = 1$ is vacuous and $k = 2$ is clear. Without loss of generality, $0 < x_k < 1$. Now rewrite $\sum_{i=1}^{k} \lambda_i x_i$ as

$$(1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i \; + \; \lambda_k x_k$$

$\square$

Compare:

- $U \subseteq \mathbb{R}^d$ is a *linear subspace* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}$.
- $U \subseteq \mathbb{R}^d$ is an *affine subspace* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}$, $\lambda + \mu = 1$.

2.1.2   Operations on convex sets

Lemma 2.3 (Operations on convex sets)**.** *Let* $K, K' \subseteq \mathbb{R}^d$ *be convex sets.*

- $K \cap K' \subseteq \mathbb{R}^d$ *is convex.*
- $K \times K' \subseteq \mathbb{R}^{d+d}$ *is convex.*
- *For any affine map* $f : \mathbb{R}^d \to \mathbb{R}^e$, $x \mapsto Ax + b$, *the image* $f(K)$ *is convex.*
- *The* Minkowski sum $K + K' := \{x + y : x \in K, y \in K'\}$ *is convex.*

Exercise 2.4. Interpret the Minkowski sum as the image of an affine map applied to a product.

Lemma 2.5. *Hyperplanes* $H = \{x \in \mathbb{R}^d : a^t x = \alpha\}$ *are convex.*
Open halfspaces $H^+ = \{x \in \mathbb{R}^d : a^t x > \alpha\}$ *and* $H^- = \{x \in \mathbb{R}^d : a^t x < \alpha\}$ *are convex.*
Closed halfspaces $\overline{H}^+ = \{x \in \mathbb{R}^d : a^t x \geq \alpha\}$ *and* $\overline{H}^- = \{x \in \mathbb{R}^d : a^t x \leq \alpha\}$ *are convex.*

More generally, for $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$,

- $\{x \in \mathbb{R}^d : Ax = 0\}$ is a linear subspace,
- $\{x \in \mathbb{R}^d : Ax = b\}$ is an affine subspace,
- $\{x \in \mathbb{R}^d : Ax < b\}$ and $\{x \in \mathbb{R}^d : Ax \leq b\}$ are convex subsets of $\mathbb{R}^d$.

2.1.3 Convex hulls, Radon's lemma and Helly's theorem

Definition 2.6 (convex hull). For any $S \subseteq \mathbb{R}^d$, the *convex hull* of S is defined as

$$\operatorname{conv}(S) := \bigcap \left\{ K \subseteq \mathbb{R}^d : K \text{ convex}, \ S \subseteq K \subseteq \mathbb{R}^d \right\}.$$

Note the analogy to the usual definition of *affine hull* (an affine subspace) and *linear hull* (or *span*), a vector subspace.

Exercise 2.7. Show that
- $\operatorname{conv}(S)$ is convex,
- $S \subseteq \operatorname{conv}(S)$,
- $S \subseteq S'$ implies $\operatorname{conv}(S) \subseteq \operatorname{conv}(S')$,
- $\operatorname{conv}(S) = S$ if S is convex, and
- $\operatorname{conv}(\operatorname{conv}(S)) = \operatorname{conv}(S)$.

Lemma 2.8 (Radon's[1] lemma). *Any $d + 2$ points $p_1, \ldots, p_{d+2} \in \mathbb{R}^d$ can be partitioned into two groups $(p_i)_i \in I$ and $(p_i)_i \notin I$ whose convex hulls intersect.*

Proof. The $d + 2$ vectors $\binom{p_1}{1}, \ldots, \binom{p_{d+2}}{1} \in \mathbb{R}^{d+1}$ are linearly dependent,

$$\lambda_1 \binom{p_1}{1} + \cdots + \lambda_{d+2} \binom{p_{d+2}}{1} = \binom{0}{0}.$$

Here not all λ_i's are zero, so some are positive, some are negative, and we can take $I := \{i : \lambda_i > 0\} \neq \emptyset$. Thus with $\Lambda := \sum_{i \in I} \lambda_i > 0$ we can rewrite the above equation as

$$\sum_{i \in I} \frac{\lambda_i}{\Lambda} p_i = \sum_{i \notin I} \frac{-\lambda_i}{\Lambda} p_i.$$

$\square$

Note that even more so Radon's lemma holds for any $n \geq d + 2$ points in $\mathbb{R}^d$.

Theorem 2.9 (Helly's Theorem). *Let $C_1, \ldots, C_N$ be a finite family of $N \geq d + 1$ convex sets such that any $d + 1$ of them have a non-empty intersection. Then the intersection of all N of them is non-empty as well.*

Proof. This is trivial for $N = d + 1$. Assume $N \geq d + 2$. We use induction on N.

By induction, for each i there is a point $\bar{p}_i$ that lies in all C_j except for possibly C_i. Now form a Radon partition of the points $\bar{p}_i$, and let p be a corresponding intersection point. About this point we find that on the one hand it lies in all C_i except for possibly those with $i \in I$, and on the other hand it lies in all C_i except for possibly those with $i \notin I$. $\square$

Note that the claim of Helly's theorem does not follow if we only require that any d sets intersect (take the C_i to be hyperplanes in general position!) or if we admit infinitely many convex sets (take $C_i := [i, \infty)$).

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[1] In class, I called this Carathéodory's lemma, which was wrong – Carathéodory's lemma is a related result, which you will see on the problem set.

2.1.4 Separation theorems and supporting hyperplanes

Definition 2.10. A hyperplane H is a *supporting hyperplane* for a convex set K if $K \subset \bar{H}^+$ and $\bar{K} \cap H \neq \emptyset$.

Theorem 2.11 (Separation Theorem)**.** *If $K, K' \neq \emptyset$ are disjoint closed convex sets, where K is compact, then there is a "separating hyperplane" H with $K \subset H^+$ and $K' \subset H^-$.*

Also, in the same situation there is a supporting hyperplane M with $K \subset \overline{M}^+$, $K \cap M \neq \emptyset$, and $K' \subset M^-$.

Proof. Define $\delta := \min\{\|p - q\| : p \in K, \ q \in K'\}$.

The minimum exists, and $\delta > 0$, due to compactness, if we replace K' by an intersection $K' \cap M \cdot B^d$ with a large ball, which does not change the result of the minimization.

Furthermore, by compactness there are $p_0 \in K$ and $q_0 \in K'$ with $\|p_0 - q_0\| = \delta$.
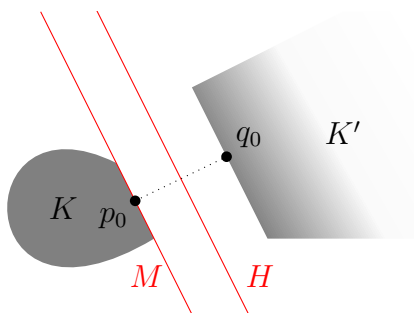
Now define H and M' by

$$H := \{x \in \mathbb{R}^d : (p_0 - q_0)^t x = (p_0 - q_0)^t (\tfrac{1}{2}p_0 + \tfrac{1}{2}q_0)\}$$

and

$$M := \{x \in \mathbb{R}^d : (p_0 - q_0)^t x = (p_0 - q_0)^t p_0\}$$

and compute. □

Example 2.12. Consider the (disjoint, closed) convex sets $K := \{(x,y) \in \mathbb{R}^2 : y \leq 0\}$ and $K' := \{(x,y) \in \mathbb{R}^2 : y \geq e^x\}$.

Separation theorems like this are extremely useful not only in Discrete Geometry (as we will see shortly), but also in Optimization. Siehe auch den Hahn–Banach Satz in der Funktionalanalysis.

2.2 Polytopes

Definition 2.13 (Polytope)**.** A *polytope* is the convex hull of a finite set, that is, a subset of the form $P = \mathrm{conv}(S) \subseteq \mathbb{R}^d$ for some finite set $S \subseteq \mathbb{R}^d$.

Examples 2.14. Polytopes: The empty set, any point, any bounded line segment, any triangle, and any convex polygon (in some $\mathbb{R}^n$) is a polytope.

Definition 2.15 (Simplex)**.** Any convex hull of a set of $k+1$ affinely independent points (in $\mathbb{R}^n$, $k \leq n$), is a *simplex*.

Lemma 2.16. *For $p_1, \ldots, p_n \in \mathbb{R}^d$, we have*

$$\mathrm{conv}(\{p_1, \ldots, p_n\}) = \{\lambda_1 p_1 + \cdots + \lambda_n p_n : \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \ \lambda_1, \ldots, \lambda_n \geq 0, \ \lambda_1 + \cdots + \lambda_n = 1\}.$$

Proof. For "$\subseteq$" we note that the RHS contains $p_1, \ldots, p_n$, and it is convex.
On the other hand, "$\supseteq$" follows from Lemma 2.2. $\qquad\square$

Definition 2.17 (Standard simplex)**.** The $(n-1)$-dimensional *standard simplex* in $\mathbb{R}^n$ is

$$\begin{aligned}
\Delta_{n-1} &= \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n, \ \lambda_1, \ldots, \lambda_n \geq 0, \ \lambda_1 + \cdots + \lambda_n = 1\} \\
&= \mathrm{conv}\{e_1, \ldots, e_n\}.
\end{aligned}$$

Corollary 2.18. *The polytopes are exactly the affine images of the standard simplices.*

Proof. … under the linear (!) map given by $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 p_1 + \cdots + \lambda_n p_n$. $\qquad\square$

Definition 2.19 (Dimension)**.** The *dimension* of a polytope (and more generally, of a convex set) is defined as the dimension of its affine hull.

Lemma 2.20. *The dimension of* $\mathrm{conv}(\{p_1, \ldots, p_n\})$ *is* $\mathrm{rank}\begin{pmatrix} p_1 & \cdots & p_n \\ 1 & \cdots & 1 \end{pmatrix} - 1$.

2.2.1 Faces

We are interested in the boundary structure of convex polytopes, as we can describe it in terms of vertices, edges, etc.

Definition 2.21 (Faces)**.** A *face* of a convex polytope P is any subset of the form $F = \{x \in P : a^t x = \alpha\}$, where the linear inequality $a^t x \leq \alpha$ is valid for P (that is, it holds for all $x \in P$).

Thus the empty set $\emptyset$ and the polytope P itself are faces, the *trivial faces*. All other faces are known as the *non-trivial faces*.

Lemma 2.22. *The non-trivial faces F of P are of the form $F = P \cap H$, where H is a supporting hyperplane of P.*

Lemma 2.23. *Every face of a polytope is a polytope.*

Proof. Let $P := \mathrm{conv}(S)$ be a polytope and let F be a face of P defined by the inequality $a^t x \leq \alpha$. Define $S_0 := \{p \in S : a^t p = \alpha\}$ and $S_- := \{p \in S : a^t p < \alpha\}$. Then $S = S_0 \cup S_-$. Now a simple calculation shows that $F = \mathrm{conv}(S_0)$: The convex combination $\lambda_1 p_1 + \cdots + \lambda_n p_n$ satisfies the inequality with equality if and only if $\lambda_i = 0$ for all $p_i \in S_-$. To see this, write for example $S_- = \{p_1, \ldots, p_k\}$ and $S_0 = \{p'_1, \ldots, p'_\ell\}$, and calculate for $x \in F$:

$$\begin{align}
\alpha = a^t x &= a^t((\lambda_1 p_1 + \cdots + \lambda_k p_k) + (\lambda'_1 p'_1 + \ldots \lambda'_\ell p'_\ell)) \tag{1} \\
&= (\lambda_1 a^t p_1 + \cdots + \lambda_k a^t p_k) + (\lambda'_1 a^t p'_1 + \ldots \lambda'_\ell a^t p'_\ell)) \tag{2} \\
&\leq (\lambda_1 \alpha + \cdots + \lambda_k \alpha) + (\lambda'_1 \alpha + \ldots \lambda'_\ell \alpha) \tag{3} \\
&= \alpha(\lambda_1 + \cdots + \lambda_k + \lambda'_1 + \ldots \lambda'_\ell) = \alpha, \tag{4}
\end{align}$$

where $\lambda_i a^t p_i \leq \lambda_i \alpha$ for $1 \leq i \leq k$ and $\lambda'_j a^t p'_j = \lambda'_j \alpha$ for $1 \leq j \leq \ell$. For this to hold, we must have $\lambda_i a^t p_i = \lambda_i \alpha$, but this holds only if $\lambda_i = 0$ for all i. Thus we have $x = \lambda'_1 p'_1 + \ldots \lambda'_\ell p'_\ell$, so $x \in \mathrm{conv}(S_0)$. $\qquad\square$

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Definition 2.24. Let P be a polytope of dimension d.

The 0-dimensional faces are called *vertices*.

The 1-dimensional faces are called *edges*.

The $(d-2)$-dimensional faces are called *ridges*.

The $(d-1)$-dimensional faces are called *facets*.

A k-dimensional face will also be called a k-face.

The set of all vertices of P is called the *vertex set* of P, denoted $V(P)$.

Proposition 2.25. *Every polytope is the convex hull of its vertex set, $P = \text{conv}(V(P))$.*

Moreover, if $P = \text{conv}(S)$, then $V(P) \subseteq S$. In particular, every polytope has finitely many vertices.

Proof. Let $P = \text{conv}(S)$ and replace S by an inclusion-minimal subset $V = V(P)$ with the property that $P = \text{conv}(V)$. Thus none of the points $p \in V$ are contained in the convex hull of the others, that is, $p \notin \text{conv}(V \backslash p)$. Now the Separation Theorem 2.11, applied to the convex sets $\{p\}$ and $\text{conv}(V \backslash p)$, implies that there is a supporting hyperplane for $\{p\}$ (that is, a hyperplane through p) which does not meet $\text{conv}(V \backslash p)$.

We take the corresponding linear inequality, which is satisfied by p with equality, and by all points in $\text{conv}(V \backslash p)$ strictly. Thus $\{p\}$ is a face: a vertex. $\qquad\square$

Proposition 2.26. *Every face of a face of P is a face of P.*

Proof. Let $F \subset P$ be a face, defined by $a^t x \leq \alpha$. Let $G \subset F$ be a face, defined by $b^t x \leq \beta$. Then for sufficiently small $\varepsilon > 0$, the inequality

$$(a + \varepsilon b)^t x \leq \alpha + \varepsilon \beta$$

is strictly satisfied for all vertices in $V(P) \backslash F$, since this is strictly satisfied for $\varepsilon = 0$, so this leads to finitely-many conditions for ε to be "small enough." It is also strictly satisfied on $F \backslash G$ if $\varepsilon > 0$, and it is satisfied with equality on G.
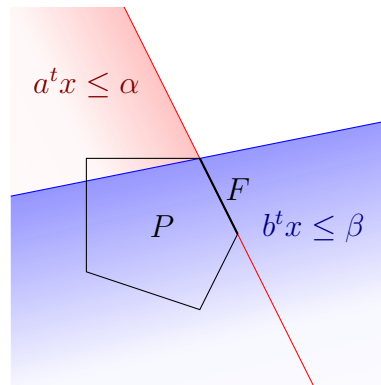
Now let x be any point in $P \backslash F$. Then we can write x as a convex combination of the vertices in P, say

$$x = (\lambda_1 v_1 + \cdots + \lambda_k v_k) + (\lambda'_1 v'_1 + \ldots \lambda'_\ell v'_\ell)$$

for $S_- = \{v_1, \ldots, v_k\}$ and $S_0 = \{v'_1, \ldots, v'_\ell\}$ as in the proof of Lemma 2.23. As x does not lie in F, the coefficient of at least one vertex v_i of P not in F is positive. This implies that the inequality displayed above is strict for x. $\qquad\square$

Corollary 2.27. *Every face F of a polytope P is the convex hull of the vertices of P that are contained in F:*
$$V(F) = F \cap V(P).$$

Proof. "$\subseteq$" is from Proposition 2.26. "$\supseteq$" is trivial. $\qquad\square$

In particular, any polytope has only finitely many faces.

Lemma 2.28. *Any intersection of faces of a polytope P is a face of P.*

Proof. Add the inequalities. $\qquad\square$

Definition 2.29 (Vertex figure). Let v be a vertex of a d-dimensional polytope P, and let H be a hyperplane that separates v from $\mathrm{conv}(V(P) \setminus \{v\})$. Then

$$P/v := P \cap H$$

is called a *vertex figure* of P at v.

Proposition 2.30. *If $P = \mathrm{conv}(S \cup \{v\})$ with $a^t v > \alpha$ while $a^s < \alpha$ for $s \in S$, where $H = \{x \in \mathbb{R}^d : a^t = \alpha\}$, then*

$$P/v = \mathrm{conv}\Big\{ \frac{a^t v - \alpha}{a^t v - a^t s} s + \frac{\alpha - a^t s}{a^t v - a^t s} v : s \in S \Big\}.$$

In particular, P/v is a polytope.

Proof. "$\supseteq$": the points $\bar{s} := \frac{a^t v - \alpha}{a^t v - a^t s} s + \frac{\alpha - a^t s}{a^t v - a^t s} v$ have been constructed as points $\lambda s + (1-\lambda)v$ such that $a^t \bar{s} = \alpha$, so $\bar{s} \in P/v$.

"$\subseteq$": *calculate* that if $x \in \mathrm{conv}(S \cup \{v\})$ satisfies $a^t x = \alpha$, then it can be written as a convex combination of the points $\bar{s}$. For this, write

$$
\begin{aligned}
x &= \sum_i \lambda_i s_i + \lambda_0 v \\
&= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \frac{a^t v - \alpha}{a^t v - a^t s_i} s_i + \lambda_0 v \\
&= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \Big(\frac{a^t v - \alpha}{a^t v - a^t s_i} s_i + \frac{\alpha - a^t s_i}{a^t v - a^t s_i} v \Big) + \Big(\lambda_0 - \sum_i \lambda_i \frac{\alpha - a^t s_i}{a^t v - \alpha} \Big) v \\
&= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \bar{s}_i + \Big(\lambda_0 - \frac{\alpha \sum_i \lambda_i - \sum_i \lambda_i a^t s_i}{a^t v - \alpha} \Big) v.
\end{aligned}
$$

At this point we use that $x \in H$, that is, $a^t x = \sum_i \lambda_i a^t s_i + \lambda_0 a^t v = \alpha$, and that this was a convex combination, so $\sum_i \lambda_i = 1 - \lambda_0$, to conclude that the last term in large parentheses is 0. $\qquad\square$

Exercise 2.31. Let $P := \mathrm{conv}\{\pi(\pm 1, \pm 1, 0, 0) : \pi \in \mathfrak{S}_4\}$ be the convex hull of all the vectors that have two ± 1 entries and two zero coordinates.

- How many vectors are these?
- Why are they all vertices?
- Why do they all have the same vertex figure?
- Compute one vertex figure.

Proposition 2.32. *For any vertex v of a d-polytope P, the k-dimensional faces of P/v are in an inclusion-preserving bijection with the $(k+1)$-dimensional faces of P that contain v.*

In particular, P/v is a polytope of dimension $d-1$.

Proof. Clearly if F is a face of P, then $F \cap H$ is a face of $P \cap H = P/v$.

Note that $v \notin H$. Thus every $(k+1)$-face $F \subseteq P$ with $v \in P$ defines a k-face F/v of P/v: From the previous proof we can see that $\mathrm{aff}((F \cap H) \cup \{v\}) = \mathrm{aff}(F)$.

For the converse, let $G \subseteq P/v$ be a k-face, defined by the inequality $b^t x \leq \beta$. Then we calculate that this inequality, plus a suitable (not necessarily positive!) multiple of the equation $a^t x = \alpha$ defining H, is satisfied with equality on $P \cap (\mathrm{aff}(G \cup \{v\}))$, but strictly on all other points of P. Explicitly, the inequality we consider is

$$(b^t + \mu a^t)x \leq \beta + \mu\alpha, \tag{5}$$

and this will be satisfied with equality on v if $(b^t + \mu a^t)v = \beta + \mu\alpha$, that is, if $\mu = -\frac{b^t v - \beta}{a^t v - \alpha}$, where the denominator is positive. This inequality (5) is valid on P/v and valid with equality on v. Let $P = \mathrm{conv}(S \cup \{v\}$. Then the inequality is valid on all points of S as well, since a point $s \in S$ that violates it would give rise to $\bar{s} \in P/v$ that violates it as well.

Thus

$$\widehat{G} := P \cap (\mathrm{aff}(G \cup \{v\}))$$

is the desired $(k+1)$-face of P. $\qquad\square$

2.2.2 Order theory and the face lattice

Definition 2.33 (Posets and lattices). A *poset* is a partially ordered set, that is, a set S with a binary relation "$\leq$" that is *reflexive* ($x \leq x$ for all $x \in S$), *asymmetric* ($x \leq y \leq x$ implies $x = y$) and *transitive* ($x \leq y \leq z$ implies $x \leq z$). (All posets we consider are finite.) Formally, the poset could be written $(S, \leq)$, but it is customary to write the same letter S for the poset.

An *interval* in a poset $(S, \leq)$ is a subposet (i.e., a subset with the induced partial order) of the form

$$[x, y] := (\{z \in S : x \leq z \leq y\}, \leq)$$

for $x, y \in S$, $x \leq y$.

A *chain* in a poset is a totally-ordered subset.

A poset is *bounded* if it has a unique minimal element, denoted $\hat{0}$, and a unique maximal element, denoted $\hat{1}$.

A poset is *graded* if it has a unique minimal element $\hat{0}$, and if for every element x of the poset, all maximal chains from $\hat{0}$ to x have the same length, called the *rank* of the element, usually

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denoted $r(x)$. The function $r : S \to \mathbb{N}_0$ is then called the *rank function* of S. If a poset is graded and has a maximal element $\hat{1}$, we write $r(S) := r(\hat{1})$ for the *rank of the poset*.

A poset is a *lattice* if any two elements a, b have a unique minimal upper bound, denoted $a \vee b$, called the *join* of a and b, and a unique maximal lower bound, denoted $a \wedge b$, and called the *meet* of a and b.

Exercise 2.34. Let $(Q, \leq)$ be a finite partial order. Show that any two of the following properties yield the third:

1. The poset is bounded.
2. Meets exist.
3. Joins exist.

Exercise 2.35. Let Q be a finite lattice, and A be an arbitrary subset. Then A has a unique minimal upper bound, the *join* $\bigvee A$, and a unique maximal lower bound, the *meet* $\bigwedge A$.

Theorem 2.36 (The polytope face lattice). *The face poset* $(\mathcal{F}, \subset)$ *of any polytope is a finite graded lattice, denoted* $L = L(P)$, *of rank* $r(L(P)) = \dim(P) + 1$.

Proof. This is a finite bounded poset, with minimal element $\hat{0} = \emptyset$ and maximal element $\hat{1} = P$. Meet exists, as clearly $F \wedge F' = F \cap F'$ is the largest face contained in both F and F'. (The intersection is a face by Lemma 2.28.) Thus $L(P)$ is a lattice.

If $G \subset F$ are faces, then in particular G is a face of F, and thus $\dim(G) < \dim(F)$. Thus all we have to prove is that if $\dim(F) \geq \dim(G) + 2$, then there is a face H with $G \subset H \subset F$.

If $F \subset P$, then $\dim(F) < \dim(P)$, so we are done by induction.

If $\emptyset \subset G$, then G has a vertex v, and $[G, F] \subseteq [v, P] = L(P/v)$, where $\dim(P/v) < \dim(P)$, so we are done by induction.

If $G = \emptyset$ and $F = P$, where $\dim(P) \geq 1$, then P has a vertex w, where $\emptyset \subset \{w\} \subset P$. $\square$

Definition 2.37 (Combinatorially equivalent). Two polytopes P and P' are *combinatorially equivalent* if their face lattices L, L' are isomorphic as posets, that is, if there is a bijection $f : L \to L'$ such that $x \leq_L y$ holds in P if and only if $f(x) \leq_{L'} f(y)$ holds in P'.

Exercise 2.38. Define "isomorphic" for posets, and for lattices. Show that if Q is a poset and L is a lattice, and if Q and L are isomorphic as posets, then Q is a lattice, and Q and L are also isomorphic as lattices.

Exercise 2.39. Let us consider the poset $D(n)$ of all divisors of the natural number n (examples to try: 24 and 42 and 64), ordered by divisibility. Are these posets? Are they bounded? Are they lattices? Graded? What is the rank function? Can you describe join and meet?

For which n is there a polytope with $D(n)$ isomorphic to its face lattice?

Lemma 2.40. *If two polytopes* P, P' *are affinely isomorphic (that is, if there is an affine bijective map* $P \to P'$), *then they are combinatorially equivalent. The converse is wrong.*

Lemma 2.41 (Face lattice of a simplex). *Let* Δ_{k-1} *be a* $(k-1)$-*dimensional simplex (with* k *vertices). Its face lattice is isomorphic to the poset of all subsets of a* k-*element set, ordered by inclusion known as the* Boolean algebra B_k *of rank* k, *as given for example by* $(2^{[k]}, \subseteq)$, *where* $2^{[k]}$ *denotes the collection of all subsets of* $[k] := \{1, 2, \ldots, k\}$.

Proof. Any two $(k-1)$-simplices are affinely equivalent.

Any subset of the vertex set of a simplex defines a face, which is a simplex. □

Exercise 2.42. Prove that if any subset of vertex set of a polytope defines a face, then the polytope is a simplex.

Theorem 2.43 (Intervals in polytope face lattices)**.** *Let $G \subseteq F$ be faces of a polytope P. Then the interval*

$$[G, F] = (\{H \in L(P) : G \subseteq H \subseteq F\}, \subseteq)$$

of $L(P)$ is the face lattice of a polytope of dimension $\dim(F) - \dim(G) - 1$.

In particular, if $G = \emptyset$, then $[G, F] = L(F)$.

In particular, if $F = P$ and $G = \{v\}$ is a vertex, then $[G, F] = L(P/v)$.

Proof. The two "in particular" statements follow from Propositions 2.26 and 2.32. Now we can use induction. □

Corollary 2.44 (Diamond property)**.** *Any interval $[x, y]$ of length 2 in a polytope face lattice contains exactly two elements z with $x < z < y$.*

This "harmless lemma" has substantial consequences.

Corollary 2.45. *For every polytope, every face is the minimal face containing a certain set of vertices. (More precisely, every face is the convex hull of the vertices it contains.)*

Simultaneously, every face is an intersection of facets (it is the intersection of the facets it is contained in).

Proof. This says that every element in the face lattice of a polytope is a join of vertices, and a meet of facets.

This can be phrased and proved entirely in lattice-theoretic language: Take a graded lattice of rank $d + 1$ with the diamond property. Then every element of rank $r(x) \leq d$ is a meet of elements of rank $d - 1$ (which would be called "co-atoms"). Simultaneously, every element of rank $r(x) > 0$ is a join of elements of rank 1 (which are called "atoms").

To prove this, note that for an element of rank $k \geq 2$ the diamond property shows that it is the join of two elements of rank $k - 1$, and by induction those are joins of atoms. Dually for meets of coatoms. □

2.2.3 Simple and simplicial polytopes

Definition 2.46. A polytope is *simplicial* if all its facets are simplices.

A polytope is *simple* if all its vertex figures are simplices.

Lemma 2.47. *A polytope is simplicial if all the proper lower intervals in its face lattice are boolean.*

A polytope is simple if all the proper upper intervals in its face lattice are boolean.

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Thus, in particular, to be simplicial or simple is a "combinatorial" property: It can be told from the face lattice.

Note that if the set of $n > d$ points $V \subset \mathbb{R}^d$ is "in general position" in the sense that no $d + 1$ points lie on a hyperplane, then $P = \mathrm{conv}(S)$ is a simplicial d-polytope.

Exercise 2.48. Every polytope that is both simple and simplicial is a simplex, or it has dimension 2.

2.2.4 $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes: The representation theorem

Theorem 2.49 (Minkowski–Weyl Representation Theorem). *Every d-dimensional polytope in $\mathbb{R}^d$ can be represented in the following equivalent ways:*

$\mathcal{V}$**-polytope** *The subset P is given as a convex hull of a finite set $V \subset \mathbb{R}^d$:*

$$P = \mathrm{conv}(V).$$

This representation is unique if V is the set of all vertices of P.

$\mathcal{H}$**-polytope** *The subset P is given as the set of solutions of a finite system of linear inequalities,*

$$P = \{x \in \mathbb{R}^d : Ax \leq a\}.$$

This representation is unique if the system "$Ax \leq a$" consists of one facet-defining linear inequality for each facet of P. (Uniqueness up to permutation of the inequalities, and up to taking positive multiples of the facet-defining inequalities.)

Proof. A $\mathcal{V}$-polytope is a special representation of what we have up to now called simply a polytope. The uniqueness was proven in Proposition 2.25.

Every $\mathcal{V}$-polytope is an $\mathcal{H}$-polytope:

The fact that every $\mathcal{V}$-polytope is the solution of a finite set of inequalities follows from a procedure called "Fourier–Motzkin elimination". For this let $V = (v_1, \ldots, v_n) \in \mathbb{R}^{d \times n}$. We write

$$
\begin{aligned}
P_{d+n} := \{ \tbinom{x}{\lambda} \in \mathbb{R}^{d+n} : & \ x = \lambda_1 v_1 + \cdots + \lambda_n v_n, \\
& \ \lambda_1 + \cdots + \lambda_n = 1, \\
& \ \lambda_1, \ldots, \lambda_n \geq 0 \}
\end{aligned}
$$

This $P_{d+n} \subset \mathbb{R}^{d+n}$ is clearly an $\mathcal{H}$-polytope (a bounded solution of a linear system of inequalities); indeed, it is an $(n-1)$-dimensional simplex, with vertices $\tbinom{v_i}{e_i}$. Furthermore, projection of P_{d+n} to $\mathbb{R}^d$ by "deleting the last n coordinates" yields P. Thus we simply have to show that "deleting the last coordinate" maps an $\mathcal{H}$-polytope to an $\mathcal{H}$-polytope.

For this, let $\pi : P' \to P''$, $\tbinom{x}{y} \mapsto x$ be such a projection map $x \in \mathbb{R}^m$, $y \in \mathbb{R}$, where P' is given by linear inequalities (and possibly equations). A point x lies in P'' if $\tbinom{x}{y}$ lies in P' for some y. Such an y exists if all the upper bounds for y (which are given by linear inequalites in the other coordinates) are larger or equal than all the lower bounds for y (which are given similarly). Thus

<center>"all upper bounds on y are larger or equal all lower bounds"</center>

yields a new system of inequalities that defines P''. (If there are equations fixing y, then those have to be taken in account as well, and have to be compatible with the inequalities.)

We leave the proof of the uniqueness part for later.

Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron:

For this we prove a similar statement for more general sets: Every subset $Q \subset \mathbb{R}^d$ that is given in the form

$$Q = \{x \in \mathbb{R}^d : Ax \le a\},$$

for some $A \in \mathbb{R}^{n \times d}$ and $a \in \mathbb{R}^n$, which we call an $\mathcal{H}$-*polyhedron* (not necessarily bounded!) can be written as a $\mathcal{V}$-*polyhedron*, in the form

$$Q = \operatorname{conv}(V) + \operatorname{cone}(Y),$$

where

$$\operatorname{cone}(Y) = \{\mu_1 y_1 + \cdots + \mu_m y_m : \mu_1, \ldots, \mu_m \ge 0\}$$

is a conical combination of the vectors in the finite set $Y = \{y_1, \ldots, y_m\} \subset \mathbb{R}^d$.

To prove this, we interpret the set Q as given above as the $\mathcal{H}$-polyhedron

$$\widehat{Q} = \{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{d+n} : Ax \le z\},$$

intersected with the subspace $\{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{d+n} : z = a\}$.

This $\widehat{Q}$ we write as a $\mathcal{V}$-polyhedron: It is the sum of the linear subspace $\{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{d+n} : Ax = z\}$, which has a cone basis given by the vectors $\pm \begin{pmatrix} e_i \\ a_i \end{pmatrix}$, and an orthant $\{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{d+n} : x = 0, z \ge 0\}$ spanned as a cone by unit vectors $\begin{pmatrix} 0 \\ e_j \end{pmatrix}$.

So it suffices to show that the intersection of any $\mathcal{V}$-polyhedron $\widehat{Q}$ with a hyperplane of the form $H_j := \{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{d+n} : z_j = a_j\}$ is again a $\mathcal{V}$-polyhedron. So let's consider

$$\widehat{Q} = \operatorname{conv}(W) + \operatorname{cone}(U) = \operatorname{conv}(W^+ \cup W^- \cup W^0) + \operatorname{cone}(U^+ \cup U^- \cup U^0),$$

where we have split the set W into the subsets lying above, on, or below the hyperplane H, and similarly with U with the hyperplane $H_j^0 := \{ \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{d+n} : z_j = 0\}$.

In this case we get lots of points in $\widehat{Q} \cap H_j$:

- points in W^0,
- intersections of H_j with segments between a point in W^+ and one in W^-,
- intersections of H_j with rays starting from a point in W^+ with direction in U^-, and
- intersections of H_j with rays starting from a point in W^- with direction in U^+.

Let $V_j(\widehat{Q})$ be the set of all these points. Similarly, collect the following directions in $\widehat{Q} \cap H_j^0$:

- directions in U^0, and
- directions obtained by a suitable combination of a direction in U^+ and one in U^-.

Let $R_j(\widehat{Q})$ be the set of all these directions. Then it is clear that

$$\widehat{Q} \cap H_j \supset \operatorname{cone}(V_j(\widehat{Q})) + \operatorname{cone}(R_j(\widehat{Q})).$$

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To prove that the converse inclusion "⊆" holds, we have to take any point $x \in \widehat{Q} \cap H_j$ and split it into contributions coming from the points and rays we have collected. It turns out that this is equivalent to finding a point in a given transportation polytope – a problem that you will solve for Problem Set 4. (Details for the computation omitted here. Example done in class.)

Every $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope:

Thus we have seen that any intersection of an $\mathcal{H}$-polyhedron with a coordinate subspace is also a $\mathcal{V}$-polyhedron, of the form $\mathrm{conv}(V) + \mathrm{cone}(Y)$. If the intersection is bounded, then clearly the $\mathcal{V}$-polyhedron is of the form $\mathrm{conv}(V)$, i.e., a $\mathcal{V}$-polytope. $\qquad\square$

Remark 2.50. Fourier–Motzin elimination is constructive, and not hard to implement. It is contained in software systems such as `PORTA` and `polymake`.

In particular, instead of solving for upper bounds and lower bounds in a variable we want to eliminate, we just take two inequalities $a^t x \leq \alpha$ and $b^t x \leq \beta$ where for some variable x_i the coefficient in one is positive and in the other is negative, say $a_i > 0$ and $b_i < 0$. Then the positive combination of the two inequalities

$$[(-b_i)a^t + (a_i)b^t]^t x \leq (-b_i)\alpha + (a_i)\beta$$

is also valid, and it does not involve the variable x_i any more: This is the elimination step performed by adding/combining inequalities.

However, the elimination algorithm is also badly exponential: If we are "unlucky", every step transforms a system of n inequalities into $(\frac{n}{2})^2$ inequalities. So within a few steps the number of inequalities can "explode". The result will typically contain many redundant inequalities, but these are not easy to detect.

2.2.5 Polarity/Duality

Definition 2.51. Let $K \subset \mathbb{R}^d$ be a subset. Its *polar* is

$$K^* = \{y \in \mathbb{R}^d : y^t x \leq 1 \text{ for all } x \in K\}.$$

Exercise. $K^* = \mathrm{conv}(K)^* = \mathrm{conv}(K \cup \{0\})^*$.

Exercise. Compute and draw K^* for axis parallel rectangles in the plane with opposite vertices
  (i) $(0,0)$ and $(M,1)$, for $M > 0$ large.
 (ii) $(-\varepsilon, -\varepsilon)$ and $(M,1)$, for $M > 0$ large, $\varepsilon > 0$ small.
(iii) $(\varepsilon, \varepsilon)$ and $(M,1)$, for $M > 0$ large, $\varepsilon > 0$ small.
What happens for $\varepsilon \to 0$, $M \to \infty$?

Lemma 2.52. *Let $K, L \subseteq \mathbb{R}^d$ be a closed convex set.*
  (i) $0 \in K^*$.
 (ii) K^* *is closed and convex.*
(iii) $K \subseteq L$ *implies* $K^* \supseteq L^*$.
 (iv) *If* $0 \in K$*, then* $K^{**} = K$.
  (v) *If* $0 \in K, L$*, then* $K \subseteq L$ *if and only if* $K^* \supseteq L^*$.
 (vi) K *is bounded if and only if* K^* *has* 0 *in its interior.*
(vii) K^* *is bounded if and only if* K *has* 0 *in its interior.*

Proof. Items (i), (ii) and (iii) are easy to see/calculate.

For (iv), we have $K \subseteq K^{**}$ by definition. If $z \notin K$, then as K is closed and convex, by the Separation Theorem there are a vector $y \neq 0$ and $\gamma \in \mathbb{R}$ such that $y^t x < \gamma$ holds for all $x \in K$, but not for $x = z$, that is, such that $y^t z > \gamma$. As $0 \in K$, we get $\gamma > 0$, and after possibly rescaling we may assume $\gamma = 1$. Thus we have that (1) $y^t x < 1$ holds for all $x \in K$, but (2) $y^t z > \gamma$. But the first condition says that $y \in K^*$, and thus the second one says that $z \notin K^{**}$. In other words, we have proved that $K^{**} \subseteq K$.

(iii) and (iv) together yield (v).

Also (iv) immediately implies (vi) and (vii), as K is bounded if and only if $K \subseteq B(0, R)$, where $B(0, R)$ is the ball with center 0 and radius R, for some suitably large R, and similarly K has 0 in the interior if and only if $B(0, \varepsilon)$ for a suitably small $\varepsilon > 0$. $\qquad\square$

Interestingly enough, we get a very explicit description of the polar of a polytope — assuming that we have both a $\mathcal{V}$- and an $\mathcal{H}$-representation available.

Theorem 2.53 (Polarity for polytopes). *Let P be a d-polytope in $\mathbb{R}^d$ with 0 in its interior, with*

$$P = \mathrm{conv}(V) = \{x \in \mathbb{R}^d : Ax \leq 1\}$$

with $V \in \mathbb{R}^{d \times n}$ and $A \in \mathbb{R}^{m \times d}$, that is, a convex hull of n points resp. the solution set of m inequalities.

Then the polar P^* is also a d-polytope with 0 in its interior, and

$$P^* = \mathrm{conv}(A^t) = \{y \in \mathbb{R}^d : V^t y \leq 1\}.$$

Under this correspondence, the vertices of P correspond to the facets of P^*, and vice versa. In particular, if the set V was chosen minimal (that is, the set of vertices of P) and the system "$Ax \leq 1$" was minimal, then $Ax \leq 1$ consists of exactly one facet-defining inequality for each facet of P.

Proof (Part I). For this, read "$P = \mathrm{conv}(V)$" as saying that P is the convex hull of the columns of V. At the same time, "$P = \{x \in \mathbb{R}^d : Ax \leq 1\}$" says that P is the polar of the set of columns of A^t. With this, everything follows from K^{**}, if we note that the first representation yields that P is bounded, and the second one implies that 0 is in the interior. $\qquad\square$

Exercise 2.54. For

$$P = \mathrm{conv}(V) = \{x \in \mathbb{R}^d : Ax \leq 1\} \qquad \text{and} \qquad P^* = \mathrm{conv}(A^t) = \{y \in \mathbb{R}^d : V^t y \leq 1\},$$

describe all the faces of P^* in terms of the faces of P — that is, give the $\mathcal{H}$-description of a face $F^\diamond$ of P^* in terms of the $\mathcal{V}$-description of P and F, etc.

Theorem 2.55 (Duality for polytopes). *Let P be a d-polytope in $\mathbb{R}^d$ wit 0 in the interior and let P^* be its polar, then the face lattice $L(P^*)$ is the "opposite" of $L(P)$.*

Proof. There are two ways to prove this. The "hard way" is to go via Exercise 2.54, and to describe a precise match between the faces $F \subset P$ and "corresponding" faces $F^\diamond \subseteq P*$.

The easier way goes via the following observation, which plainly says that the incidences between the vertices and the facets of a polytope already fix the combinatorial type (i.e., the face lattice). $\qquad\square$

Terminology: If $L(Q) = L(P)^{opp}$, then we say that Q is *a dual* of P. Note that every polytope has many duals, but only one polar polytope (if it has 0 in the interior etc.)

Corollary 2.56. *A polytope is P is simple if and only if P^* is simplicial, and vice versa.*

Theorem 2.57. *Let P be a d-dimensional polytope with n vertices and m facets.*
Then the combinatorial type of P (that is, the face lattice $L(P)$) is determined by the vertex–facet incidences, that is, by the matrix

$$I(P) = (\kappa_{ij}) \in \{0,1\}^{n \times m},$$

where $\kappa_{ij} = 1$ if $v_i \in F_j$, and $\kappa_{ij} = 0$ otherwise, for some arbitrary labelling $v_1, \ldots, v_n$ of the vertices and $F_1, \ldots, F_m$ of the facets.

Proof. The faces are the intersections of facets, and the vertex sets of faces are exactly the intersections of vertex sets of facets, by Corollaries 2.27 and 2.45.

Thus the vertex sets of facets are given by the rows of the matrix $I(P)$, and the vertex sets of faces are exactly the intersections of these rows, which we interpret as incidence vectors of vertex sets of facets. $\qquad\square$

Lemma 2.58 (Characterization of vertices). *Let $P = \mathrm{conv}(V) = \{x \in \mathbb{R}^d : Ax \leq 1\}$. Then $v_0 \in \mathbb{R}^d$ is a vertex of P if and only if any one of the following conditions are satisfied:*

(i) $\{v_0\}$ *is a face of dimension 0, that is, $v_0 \in P$ but there is an inequality $a^t x \leq \alpha$ such that $a^t v_0 = \alpha$, while $a^t v_i < \alpha$ for all other $v_i \in V$.*

(ii) $v_0 \in V$, *and there is an inequality $a^t x \leq 1$ such that $a^t v_0 = 1$, while $a^t v_i < 1$ for all other $v_i \in V$.*

(iii) v_0 *is a point in P such that $\{v_0\}$ is an intersection of some facets of P.*

(iv) v_0 *is a point in P such that $\{v_0\}$ is an intersection of d facet-defining hyperplanes $H_i = \{x \in \mathbb{R}^d : a_i^t x = 1\}$ $(1 \leq i \leq d)$.*

Proof. (i) is the definition of a vertex (0-dimensional face).

(ii): As 0 lies in the interior of P, the inequality from (i) has to have $\alpha > 0$, so we can rescale to get $\alpha = 1$. Also v_0 lies in V, otherwise we would have $a^t v < 1$ for all $v \in V$ and thus $a^t x < 1$ for all $x \in P$.

(iii): We know that every face (and thus every vertex) is an intersection of facets. Conversely, every intersection of facets is a face, and if the face is a single point, it is a vertex.

(iv): If v_0 is a vertex, then it is contained in a maximal chain of faces $v_0 = G_0 \subset G_1 \subset G_{d-2} \subset G_{d-1}$, where G_i is a face of dimension i and $G_i = G_{i+1} \cap F_i$, where F_i is a facet — since every face is an intersection of facets. Let $H_i = \mathrm{aff}(F_i)$, then we have that $F_i \subset H_i$ and $F_i = P \cap H_i$, and thus

$$G_i = G_{i+1} \cap F_i \subseteq G_{i+1} \cap H_i \subseteq G_{i+1} \cap P \cap H_i = G_{i+1} \cap F_i,$$

which yields $G_i = G_{i+1} \cap H_i$. We conclude that each of the intersections H_{d-1}, $H_{d-1} \cap H_{d-2}$, $H_{d-1} \cap H_{d-2} \cap \cdots \cap H_0$ strictly contains the next one — and thus the last one in the sequence has dimension 0, it is a single point, namely $G_0 = \{v_0\}$. On the other hand, v_0 is then an intersection of facets, so it is a vertex. $\qquad\square$

Lemma 2.59 (Characterization of facets). *Let $P = \mathrm{conv}(V) = \{x \in \mathbb{R}^d : Ax \leq 1\}$. Then $F \subset P$ is a facet of P if and only if any one of the following conditions are satisfied:*

   (i) *$F = \{x \in P : a^t x = \alpha\}$ for an inequality $a^t x \leq \alpha$ that is valid for all of P, with $\dim(F) = d - 1$.*

  (ii) *$F = \{x \in P : a^t x = \alpha\}$ for an inequality $a^t x \leq \alpha$ that is valid for all of P, with d affinely-independent points $v_1, \ldots, v_d$ from the set V that satisfy $a^t v_i = 1$.*

 (iii) *$F = \{x \in P : a_i^t x = 1\}$ for an inequality $a_i^t x \leq 1$ from the system $Ax \leq 1$, with d affinely-independent points $v_1, \ldots, v_d$ from the set V that satisfy $a_i^t v_j = 1$.*

Proof. (i) is the definition of a facet.

(ii): Let $V_0 \subseteq V$ be the subset of all the $v_i \in V$ that satisfy the inequality from (i) with equality. If the affine hull of these points has dimension $d - 1$, then we can choose d that span this hull. If the affine hull has smaller dimension, then we note $F \subset \mathrm{aff}(V_0)$, so F is not a facet.

(iii): Here the new information is that the facet-defining inequalities all come from the system $Ax \leq 1$. However, note that the inequality $a^t x \leq 1$ that satisfies $a^t v_j = 1$ for $1 \leq j \leq d$ is unique. If it were not in the inequality system, then the barycenter $\frac{1}{d}(v_1^+ \cdots + v_d)$ would lie in the *interior* of the set defined by $Ax \leq 1$; on the other hand, it lies on the boundary due to the inequality $a^t x \leq 1$. $\qquad\square$

Proof of Theorem 2.53 (Part II). From the characterization Lemma 2.59, we see that the facets of P are exactly given by the inequalities of the system $Ax \leq 1$, under the assumption that the system was chosen to be minimal.

The assumption that the two systems for P are minimal implies that the systems for P^* are also minimal, otherwise we would get a contradiction to $P^{**} = P$. $\qquad\square$

Proposition 2.60. *The incidence matrix $I(P)$ may be a rather compact encoding of a polytope, but it is not so easy to read things off.*

(1) *To get the dimension d of a polytope from $I(P)$ we have to find a sequence of columns such that the first column is arbitrary (corresponding to a facet) and each subsequent one is chosen to have a* maximal *intersection with the intersection of the previously-chosen ones, thus yielding the next face of a maximal chain.*

(2) *The incidence matrix of the polar is the transpose of the matrix: $I(P^*) = I(P)^t$.*

(3) *If $\dim(P) = d$, then P is simplicial if each column of $I(P)$ contains exactly d ones.*

(4) *If $\dim(P) = d$, then P is simple if each row of $I(P)$ contains exactly d ones.*

Proof of Theorem 2.55. With completing the proof of Theorem 2.53, we get that the vertices of P correspond to the facets of P^*, and vice versa. Thus the $I(P^*)$ is the transpose of $I(P)$, and thus $L(P^*)$ is the opposite of $L(P)$. $\qquad\square$

2.2.6 The Farkas lemmas

"The Farkas lemma" is a result that comes in many different flavors; it says that if something happens in polyhedral combinatorics, then there always is a concrete reason. Here is a basic version:

Proposition 2.61. *A system $Ax \leq a$ has no solution if and only if there is a vector $c \geq 0$ such that $c^t A = 0$ and $c^t a = -1$.*

Proof. The Farkas lemmas can be derived from Separation Theorems, or from each other, or from Fourier–Motzkin. We sketch Fourier–Motzkin: We can eliminate *all* the variables from the system $Ax \leq a$, such that the resulting system of inequalities $0 \leq \gamma_i$ has a solution if and only if the original system has none. Moreover, all inequalities in the resulting system are non-negative combinations of the inequalities in the original system.

Thus if the resulting system has no solution, then one inequality ready "$0 \leq \gamma_i$" for some $\gamma_i < 0$. Indeed, we may rescale to get $\gamma_i = -1$. The inequality is obtained by non-negative combination, that is, $c^t A = 0$, $c^t a = \gamma_i = -1$.

Conversely, check that the existence of c implies that the system has no solution. $\square$

3   Polytope theory

3.1   Examples, examples, examples

What do we want to know?
- dimension d
- number of vertices $f_0 = n$, number of facets f_{d-1}
- $\mathcal{V}$- and $\mathcal{H}$-description
- f-vector $(f_0, f_1, \ldots, f_{d-1})$
- graph
- simple? simplicial?
- [diameter, surface area, volume? – not so much a topic of this course]
- dual polytope?
- symmetries?
- combinatorial type? incidence matrix?
- face lattice L
- etc.

We will mix a discussion of specific (classes of) examples with a discussion of constructions – which produce new examples.

Note that the various classes of examples we describe will not be disjoint (example: every simplex is a pyramid, every cube is a prism, a triangle is both a simplex and a polygon, etc.)

3.1.1   Basic building blocks

Example 3.1 (The (regular) convex polygons). Let P be any 2-dimensional polytope, and $n = f_0(P)$ its number of vertices. Then $n \geq 3$ and $f_1(P) = n$. Any two 2-polytopes with n vertices are combinatorially equivalent — and they are in particular equivalent to the *regular convex n-gon* given by

$$P_2(n) = \operatorname{conv}\{(\cos(\tfrac{k}{n}2\pi), \sin(\tfrac{k}{n}2\pi)) : 0 \leq k \leq n\}.$$

This example in particular contains the complete classification of 2-dimensional polytopes.

Example 3.2 (The d-simplex). The standard simplex of Definition 2.17 may be described as

$$\begin{aligned}\Delta_d &= \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{d+1},\ \lambda_0, \ldots, \lambda_d \geq 0,\ \lambda_1 + \cdots + \lambda_{d+1} = 1\} \\ &= \operatorname{conv}\{e_1, \ldots, e_{d+1}\}.\end{aligned}$$

This is a d-dimensional polytope in $\mathbb{R}^{d+1}$. It has $d+1$ vertices and $d+1$ facets; the k-faces correspond to the $(k+1)$-subsets of $[d+1]$. In particular, the face lattice is a boolean algebra B_{d+1}, and we get

$$f_k(\Delta_d) = \binom{d+1}{k+1}.$$

The standard d-simplex has the symmetry group $\mathfrak{S}_{d+1}$, acting by permutation of coordinates (and thus of vertices).

A full-dimensional version of the standard simplex is obtained by deleting the last coordinate,

$$\Delta'_d = \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^{d+1}, \ \lambda_0, \ldots, \lambda_d \geq 0, \ \lambda_1 + \cdots + \lambda_d \leq 1\}$$
$$= \mathrm{conv}(0, e_1, \ldots, e_d).$$

This simplex has volume $\frac{1}{d}$. It has a smaller symmetry group than Δ_d. We leave it as an exercise to construct and describe a full-dimensional fully-symmetric realization of Δ_d.

Example 3.3 (The d-cube). Again there are two very familiar versions of the d-dimensional cube, the $0/1$-cube

$$C_d^{01} = \mathrm{conv}\{0, 1\}^d = \{x \in \mathbb{R}^d : 0 \leq x_k \leq d\} = [0, 1]^d$$

and the ± 1-cube

$$C_d = \mathrm{conv}\{1, -1\}^d = \{x \in \mathbb{R}^d : -1 \leq x_k \leq d\} = [-1, 1]^d = \{x \in \mathbb{R}^d : \|x\|_\infty \leq 1\}.$$

They are equivalent by a similarity transformation.

The non-empty k-faces are obtained by choosing k coordinates which have the full range of $[0, 1]$ resp. $[-1, 1]$ and fixing the other $d - k$ vertices to the lower or upper bound. In particular, this yields

$$f_k(C_d) = 2^{d-k} \binom{d}{k}$$

for $k \geq 0$, while $f_{-1} = 1$. In particular, the d-cube has 3^d non-empty faces.
The d-cube is simple.

The symmetry group of C_d is generated by the permutations of coordinates and by the reflections in coordinate hyperplanes. It has $2^d d!$ elements, and is known as the group of signed permutations, or as the *hyperoctahedral group*.

Exercise 3.4. For which $k = k(d)$ does the d-cube have the largest number of k-faces? To answer this, analyze the quotients f_k/f_{k-1}, and show that they decrease with k. Conclude that the f-vector of the d-cube is *unimodal*, that is,

$$f_0 < f_1 < \cdots < f_{k(d)} \geq f_{k(d)+1} > \cdots > f_{d-1}.$$

Example 3.5 (The d-dimensional crosspolytope[2]). The standard coordinates for the d-*dimensional crosspolytope* are given by

$$C_d^* = \mathrm{conv}\{\pm e_1, \ldots, \pm e_d\}$$
$$= \{x \in \mathbb{R}^d : \pm x_1 + \cdots + \pm x_d \leq 1\} = \{x \in \mathbb{R}^d : \|x\|_1 \leq 1\}.$$

The proper k-faces are obtained by choosing $k + 1$ coordinates, and a sign for each of them, so

$$f_k(C_d^*) = 2^{k+1} \binom{d}{k+1}$$

for $k < d$, while $f_d = 1$. In particular, the d-crosspolytope has 3^d non-empty faces. And indeed, this is the polar dual of the d-cube, so in particular it has the same number of faces. The d-crosspolytope is simplicial. Its symmetry group is again the hyperoctahedral group.

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[2]Compare Problem Sheet 2 (Problem 2).

Exercise 3.6 (The Half-cube). Let

$$H_d := \text{conv}\{x \in \{0,1\}^d : x_1 + \cdots + x_d \in 2\mathbb{Z}\}$$

be the dth *half-cube*.

(i) Describe H_d for $d \leq 4$.
(ii) Describe the facets of H_d: How many are they, what are their combinatorial types?
     (The cases $d = 1, 2, 3$ need to be argued separately.)
(iii) Give an $\mathcal{H}$-representation of H_d.
     (The cases $d = 1, 2, 3$ need to be argued separately.)
(iv) Show that H_d is "3-simplicial," that is, all its 3-faces are tetrahedra.
(v) Show that H_d is simplicial for $d \leq 4$, but not for $d > 4$.

3.1.2 Some basic constructions

Proposition 3.7 (Product[3]). *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes, then the product*

$$P \times Q \subset \mathbb{R}^{d+e}$$

is a polytope of dimension $\dim(P) + \dim(Q)$. *Its non-empty faces are the products of non-empty faces of P and of Q. Thus the product construction is combinatorial: the face lattice of $P \times Q$ can be derived from the face lattices of P and of Q. In particular,*

$$f_k(P \times Q) = \sum_{i=0}^{k} f_i(P) f_{k-i}(Q) \qquad \text{for } k \geq 0.$$

$P \times Q$ is simple if and only if P and Q are simple.
$P \times Q$ is never simplial, unless one of P and Q is 0-dimensional, or they are both 1-dimensional (and $P \times Q$ is a quadrilateral).

Example 3.8 (Prisms). Let $P \subset \mathbb{R}^d$ be a polytope. If I is an interval (that is, a 1-dimensional polytope, such as $I = [0,1]$ or $I = [-1,1]$), then the product $P \times I \subset \mathbb{R}^{d+1}$ is a *prism* over P. Then $\dim(P \times I) = \dim(P) + 1$. The non-empty faces of the prism $P \times I$ for $I = [0,1]$ are

- the faces of the *base* $P \times \{0\}$, which is isomorphic to P,
- the faces of the *top* $P \times \{1\}$, which is also isomorphic to P,
- the *vertical* faces of the form $P \times I$, where every non-empty k-face of P corresponds to a unique vertical $(k+1)$-face of P.

This also yields a drawing of the face lattice of P.

Note: the d-cube, $d > 0$, is an iterated prism.

Exercise 3.9. Define the *f-polynomial* of a d-polytope as

$$f_P(t) := 1 + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d + t^{d+1}.$$

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[3]Compare Problem Sheet 2 (Problem 1).

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(a) Describe the f-polynomial $f_{P \times I}$ of the prism $P \times I$ in terms of the f-polynomial of P. Deduce from this a formula for the f-polynomial of the d-cube.

(b) Describe the f-polynomial of P^* in terms of the polynomial of P. Deduce from this a formula for the f-polynomial of the d-crosspolytope.

(c) Describe the f-polynomial of $P \times Q$ in terms of the polynomials of P and of Q.

Proposition 3.10 (Direct sum[4])**.** *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes with the origin in the interior, then*

$$P \oplus Q := \mathrm{conv}(P \times \{0\} \cup \{0\} \times Q) \subset \mathbb{R}^{d+e}$$

is a polytope of dimension $\dim(P) + \dim(Q)$.
Its proper faces are all of the form $F * G := \mathrm{conv}(F \times \{0\} \cup \{0\} \times G)$, where $F \subset P$ and $G \subset Q$ are proper faces, and $\dim(F * G) = \dim(F) + \dim(G) + 1$.
In particular the direct sum is combinatorial.

Example 3.11 (Bipyramids). If P is a polytope, then $P \oplus I$ is a *bipyramid* over P: It has dimension $\dim(P) + 1$, $f_0(P) + 2$ vertices, $2 f_{\dim(P)-1}$ facets, etc.
For example, the d-crosspolytope is an (iterated) bipyramid.

Proposition 3.12. *Product and direct sum are dual constructions: If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are polytopes with the origin in the interior, then*

$$(P \times Q)^* = P^* \oplus Q^*.$$

Example 3.13 (A neighborly polytope). The direct sum $\Delta_2 \oplus \Delta_2$ [constructed from two triangles with the origin in the interior] is neighborly: This is a 4-dimensional polytope with $f_0 = 6$ vertices such that any two vertices are joined by an edge. In particular, $f_1(\Delta_2 \oplus \Delta_2) = \binom{f_0}{2} = \binom{6}{2} = 15$.

Proposition 3.14 (Joins)**.** *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes, then the* join

$$P * Q := \mathrm{conv}\left(\left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} : x \in P \right\} \cup \left\{ \begin{pmatrix} 0 \\ y \\ 1 \end{pmatrix} : y \in Q \right\}\right) \subset \mathbb{R}^{d+e+1}$$

is a polytope of dimension $\dim(P) + \dim(Q) + 1$.
Its faces are the joins of the faces of P and the faces of Q. Thus the join construction is combinatorial: the face lattice of $P * Q$ can be derived from the face lattices of P and of Q — it is simply the product,

$$L(P * Q) \cong L(P) \times L(Q).$$

In particular,

$$f_k(P * Q) = \sum_i f_i(P) f_{k-i-1}(Q) \qquad \text{for all } k.$$

$P * Q$ is neither simple nor simplicial, except if both P and Q are simplices, or if one them is empty and the other one is simple resp. simplicial.

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[4]Compare Problem Sheet 5 (Problem 1).

Example 3.15 (Pyramids). $P * \{v\}$ is the *pyramid* over P.

Remark 3.16. We are usually interested in polytopes only up to affine transformations. Thus we perform constructions such as products, direct sums, and joins in more generality than the one indicated above. Also, often P and Q lie in the same higher-dimensional vector space, and we want to see their product/join/direct sum in the same space:

- If P, Q lie in transversal affine subspaces of a real vectorspace $V \cong \mathbb{R}^N$, e.g. $P \subset V'$, $Q \subset V''$, $V' \cap V'' = \{p\}$, then

$$\{x + y : x \in P, \, y \in Q\}$$

  is (affinely equivalent to) the product of P and Q.
- If P, Q lie in transversal affine subspaces V' resp. V'' of V, where $V' \cap V''$ is a relative interior point of P and of Q, then

$$\mathrm{conv}(P \cup Q)$$

  is (affinely equivalent to) the direct sum of P and Q.
- If P, Q lie in skew subspaces of V, then

$$\mathrm{conv}(P \cup Q)$$

  is (affinely equivalent to) the join of P and Q.

Proof. ... left as an exercise. It helps to know the definitions, e.g. the following ... □

Definition (*Reminder* from Lemma 2.40: Affine equivalence, a.k.a. affinely isomorphic). Affine maps between vector spaces V and W are maps that satisfy $f(\lambda x + (1 - \lambda)y = \lambda f(x) + (1 - \lambda)f(y)$; they have the form $f(x) = Ax + b$ for a suitable matrix A and vector b.

Two polytopes $P \subset V$ and $Q \subset W$ are *affinely equivalent* if there is an affine map $f : V \to W$ such that $f(P) = Q$, where $f : P \to Q$ is a bijection. (Note that this does not require that $f : V \to W$ is a bijection – f does not need to be injective or surjective.)

Affine equivalence is an equivalence relation. In particular, affinely equivalent polytopes are combinatorially equivalent (for this, recall Lemma 2.40.)

Exercise 3.17. Show that the join construction is self-dual,

$$(P * Q)^* \cong (P^* * Q^*).$$

How do you have to interpret/adapt the notations/constructions to make this true?

Exercise 3.18. In $\mathbb{R}^d$, what is the smallest example of a polytope that is not (combinatorially equivalent to) a join, a product or a direct sum? After you have answered that: How did you interpret "smallest"?

Example 3.19 (The Hanner polytopes/The 3^d conjecture). The *Hanner polytopes* are defined as all polytopes that can be generated from $[-1, +1]$ by repeatedly applying products, direct sums, and polarity. This includes the d-dimensional cube and the d-dimensional cross polytope, but many more polytopes. (For example, a prism over an octahedron.)

All d-dimensional Hanner polytopes are centrally symmetric, and they have exactly $3^d + 1$ faces (equivalently: 3^d non-empty faces; equivalently: 3^d proper faces).

The 3^d Conjecture (by Gil Kalai, 1988 [14]) says that every centrally-symmetric d-polytope has at least $3^d + 1$ faces, and that in the case of equality it is (equivalent to) a Hanner polytope. Up to now, this is proved only for $d \le 4$; see Sanyal, Werner & Ziegler [20].

3.1.3 Stacking, and stacked polytopes

Definition 3.20 (Stacking). Let P be a polytope and F a facet. *Stacking* a pyramid onto a facet F yields a polytope

$$P' := \operatorname{conv}(P \cup \{v_0\}) = P \cup \operatorname{conv}(F \cup \{v_0\})$$

with a new vertex v_0 such that all such that all proper faces of P, except for F, are also facets of $\operatorname{conv}(P \cup \{v_0\})$

Lemma 3.21. *Let P be a d-polytope, and $F \subset P$ a facet.*
The proper faces of $P' := \operatorname{Stack}(P, F)$ are
– all proper faces of P, except for F, and
– the pyramids $\operatorname{conv}(G \cup v_0)$, for all proper faces $G \subset F$.
The f-vector of P' is hence

$$f_i(P') = \begin{cases} f_i(P) + f_{i-1}(F) & \text{for } i < d - 1 \\ f_{d-1}(P) + f_{d-2}(F) - 1 & \text{for } i = d - 1. \end{cases}$$

Definition 3.22 (Beneath/beyond). Let $P \subset \mathbb{R}^d$ be a d-polytope, and $F \subset P$ a facet.
A point $v \notin P$ lies *beneath* the facet F if v and the interior of P lie on the same side of the hyperplane H_F spanned by F.
A point $v \notin P$ lies *beyond* the facet F if v and the interior of P lie on different sides of the hyperplane H_F spanned by F.

Thus "stacking onto a facet F" describes the situation when a new point/vertex lies beyond one particular facet $F \subset P$ and beneath *all other* facets of P.

Exercise 3.23. Let $P \subset \mathbb{R}^d$ be a d-polytope, and $F \subset P$ a facet. Let $v_1, \dots, v_n$ be the vertices of P, and assume that $v_1, \dots, v_m$ for some $m < n$ are the vertices of F.
Show that
$$(1 - \lambda)\tfrac{1}{n}(v_1 + \cdots + v_n) + \lambda\tfrac{1}{m}(v_1 + \cdots + v_m)$$
– for $\lambda = 0$ is a point in the interior of P,
– for $\lambda = 1$ is a point in the relative interior of F,
– for $\lambda > 1$ is a point beyond F, which lies beneath all other facets of P if λ is small enough (but larger than 1).

Definition 3.24. A d-dimensional *stacked polytope* $\operatorname{Stack}_d(d + 1 + n)$ on $d + 1 + n$ vertices, for $n \ge 0$, is obtained from a d-simplex $\Delta_d \subset \mathbb{R}^d$ by repeating the operation "stacking onto a facet" n times.

Exercise 3.25. Show that for $d \geq 3$ and sufficiently large n, there are different combinatorial types of stacked d-polytopes on $d + 1 + n$ vertices.

Discuss how the combinatorial type of $\mathrm{Stack}_d(d + 1 + n)$ can be described in terms of a (graph theoretical) tree. Do different trees describe different polytopes? Do different stacked polytopes have different trees?

Use this to estimate the number of different stacked polytopes $\mathrm{Stack}_d(d + 1 + n)$ for some fixed $d \geq 3$ and large n.

Proposition 3.26. *The f-vector of* $\mathrm{Stack}_d(d + 1 + n)$ *is*

$$ f_i(\mathrm{Stack}_d(d + 1 + n)) = \begin{cases} \binom{d+1}{i+1} + n\binom{d}{i} & \text{for } i < d - 1 \\ d + 1 + n(d - 1) & \text{for } i = d - 1. \end{cases} $$

Exercise 3.27. Compute and sketch the f-vector of the stacked polytope $\mathrm{Stack}_{10}(42)$. In particular, how many facets does it have? Which is the largest entry of the f-vector?

Proposition 3.28. *The stacked polytope* $\mathrm{Stack}_3(8)$ *obtained by stacking onto all* 4 *facets of a tetrahedron cannot be realized with all vertices on a sphere, so it is not* inscribable.

Proof. Stereographic projection from a tetrahedron vertex, and then an angle count in the resulting *Delaunay triangulation*. See Gonska & Ziegler [11]. (Delaunay triangulations will be discussed later.) $\square$

3.1.4 Cyclic polytopes

Definition 3.29 (The moment curve). The *moment curve* in $\mathbb{R}^d$ is the monomial curve

$$ \gamma : \mathbb{R} \longrightarrow \mathbb{R}^d, \quad t \to (t, t^2, \ldots, t^d)^T. $$

For $d = 2$, this yields the standard parabola.

Lemma 3.30. *The moment curve has degree d: Every hyperplane in $\mathbb{R}^d$ cuts the moment curve in at most d points.*

(Hyperplanes cutting the curve in exactly d points exist.)

Proof 1. If the hyperplane is given by $a_1 x_1 + \cdots + a_d x_d = -a_0$, where one of the a_i $(i \geq 1)$ is not zero, then the intersection points are exactly the roots of the non-constant (!) polynomial $a_0 + a_1 t + \cdots + a_d t^d$. $\square$

Proof 2. Any $d+1$ points on the curve span a d-simplex: Indeed, the corresponding determinant (which computes the volume of the simplex) is a Vandermonde determinant. $\square$

Definition 3.31 (Cyclic polytopes). Let $n > d > 1$ and $t_1 < \cdots < t_n$. The *cyclic polytope* $C_d(t_1, \ldots, t_n)$ is defined as the convex hull of n points on the moment curve:

$$ C_d(t_1, \ldots, t_n) := \mathrm{conv}\{\gamma(t_1), \ldots, \gamma(t_n)\}. $$

For example, $C_2(t_1, \ldots, t_n)$ is the convex hull of n points on the standard parabola.

Lemma 3.32. *For any $n > d \geq 2$ and $t_1 < \cdots < t_n$ the cyclic polytope $C_d(t_1, \ldots, t_n)$ is d-dimensional, simplicial, with n vertices.*

Proof. It is d-dimensional since any $d + 1$ of the points $\gamma(t_i)$ are affinely independent. For the same reason it is simplicial. That each point $\gamma(t_i)$ defines a vertex can be seen from the projection to the first two coordinates, which yields $C_2(t_1, \ldots, t_n)$. $\square$

Definition 3.33 (Neighborly)**.** A d-polytope is called k-*neighborly* if any k vertices form the vertices of a $(k-1)$-face. The $\lfloor d/2 \rfloor$-neighborly polytopes are simply called *neighborly*.

Proposition 3.34. *The cyclic polytopes are neighborly.*

In particular, any two vertices of $C_4(n)$ are joined by an edge (1-face), so $C_4(n)$ has a "complete graph".

Proof. A hyperplane $a_0 + a_1 x_1 + \cdots + a_d x_d = 0$ is a supporting hyperplane for $C_d(t_1, \ldots, t_n)$ if and only if the polynomial $f(t) = a_0 + a_1 t + \cdots + a_d t^d = 0$ vanishes at some of the t_i's and has always the same sign at the others. Besides, $\operatorname{conv}\{\gamma(t_i) \mid f(t_i) = 0\}$ is then a face of $C_d(t_1, \ldots, t_n)$.

Let $I \subset \{1, \ldots, n\}$ be such that $|I| \leq \lfloor d/2 \rfloor$. Then the polynomial $f(t) := \prod_{i \in I}(t - t_i)^2$ vanishes at t_i for $i \in I$, is positive on t_j for $j \notin I$, and has $\deg f \leq n$. Thus it yields us a supporting hyperplane through all of $\gamma(t_i)$ with $i \in I$. $\square$

Exercise 3.35. Prove that any $(\lfloor d/2 \rfloor + 1)$-neighborly polytope is a simplex.
Hint: Use Radon's lemma.

Remark 3.36. There are neighborly non-cyclic polytopes (combinatorially different from any cyclic polytope). In dimension 4, the smallest examples have 8 vertices.

Convince yourself that a cyclic 3-polytope looks like this:
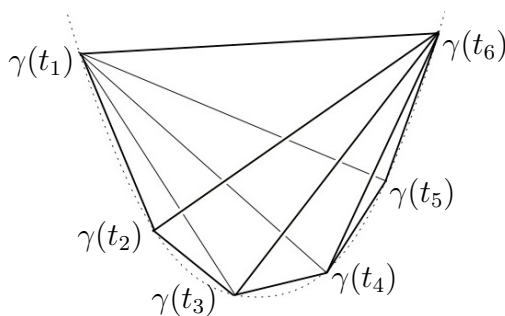
Proposition 3.37 (Gale's evenness criterion)**.** *The facets of a cyclic polytope $C_d(t_1, \ldots, t_n)$ are given by the simplices $F(\{i_1, \ldots, i_d\}) = \operatorname{conv}\{\gamma(t_{i_1}), \ldots, \gamma(t_{i_d})\}$ such that $I = \{i_1, \ldots, i_d\}$ has the following evenness property: For any two i, j taken from $\{1, \ldots, n\}$ that do* not *lie in I, there is an* even *number of i_k lying between them.*

For example, the facets of $C_3(6)$, as depicted above, are 123, 134, 145, 156, 126, 236, 346, 456.

Proof. By Lemma 3.30, the points $\gamma(t_{i_1}), \ldots, \gamma(t_{i_d})$ span a hyperplane H, and the moment curve passes not only touches, but passes through the hyperplane H at each of the points. Thus the evenness condition exactly guarantees that all other points $\gamma(t_i), \gamma(t_j)$ lie on the same side of H. □

Corollary 3.38. *The construction of the cyclic polytopes in Definition 3.31 is combinatorial, that is, the combinatorial type of $C_d(t_1, \ldots, t_n)$ does not depend on the specific values of $t_1, \ldots, t_n$, but only on the parameters d and n. This justifies the notation $C_d(n)$ for the combinatorial type of $C_d(t_1, \ldots, t_n)$*

Exercise 3.39. Show that $C_d(d+1)$ is a d-simplex, and $C_d(d+2)$ is a direct sum $\Delta_{\lceil d/2 \rceil} \oplus \Delta_{\lfloor d/2 \rfloor}$.

Proposition 3.40. *The number of facets of $C_d(n)$ is*

$$f_{d-1}(C_d(n)) = \begin{cases} \binom{n-e}{e} + \binom{n-e-1}{e-1} & \textit{if } d = 2e, \\ 2\binom{n-1-e}{e} & \textit{if } d = 2e+1. \end{cases}$$

Proof. Here is a different interpretation of the index set of a facet: It consists of $\lfloor d/2 \rfloor$ disjoint pairs $\{i, i+1\}$, plus possibly the singletons 1 or n.

Indeed, the decomposition into pairs and singletons is unique. Each facet includes exactly one of the singletons if d is odd, and both or none of them if d is even.

Now a simple bijection (remove the second entry from each pair, and renumber) shows that each of the singleton cases can be counted by a simple binomial coefficient:

$d = 2e$ even, no singleton: $\binom{n-e}{e}$ facets.

$d = 2e$ even, both singletons: $\binom{(n-2)-(e-1)}{e-1}$ facets.

$d = 2e+1$, singleton n: $\binom{n-1-e}{e}$ facets. □

Example 3.41. The facets of $C_4(8)$ are
1234, 1245, 1256, 1267, 1278, 2345, 2356, 2367, 2378, 3456, 3467, 3478, 4567, 4578, 5678, 1238, 1348, 1458, 1568, 1678.
Note that this is 20 facets: more than the 16 facets of the 4-dimensional cross polytope, which is also simplicial and has the same number of vertices.

Exercise 3.42. Find (in the literature, or by figuring it out) a formula for all the f-numbers $f_k(C_d(n))$. Use it to compute and sketch the f-vector of the cyclic polytope $C_{10}(42)$. In particular, how many facets does it have? Which is the largest entry of the f-vector?

Exercise 3.43. Show that for $n > d > 1$ the following are equivalent:
  (i) $C_d(n)$ has a simple vertex (i.e., a vertex that is contained in exactly d facets),
  (ii) $C_d(n)$ is a stacked polytope,
  (iii) $d \leq 3$ or $n \leq d+2$.

3.1.5 A quote

> The whole is greater than the part, but it is also greater than the sum of its parts. There is no need, then, to be overly obsessed with limited and particular questions. We constantly have to broaden our horizons and see the greater good which will benefit us all. [...]

> Here our model is not the sphere, which is no greater than its parts, where every point is equidistant from the centre, and there are no differences between them. Instead, it is the polyhedron, which reflects the convergence of all its parts, each of which preserves its distinctiveness. Pastoral and political activity alike seek to gather in this polyhedron the best of each. There is a place for the poor and their culture, their aspirations and their potential. Even people who can be considered dubious on account of their errors have something to offer which must not be overlooked. It is the convergence of peoples who, within the universal order, maintain their own individuality; it is the sum total of persons within a society which pursues the common good, which truly has a place for everyone.

(Pope Francis, November 2013 [17])

3.1.6 Combinatorial optimization and 0/1-Polytopes

Why 0/1-polytopes are interesting, important, remarkable, complicated.

Definition 3.44 (0/1-polytopes). A *0/1-polytope* is subpolytope of the $0/1$-cube, that is,

$$P = \operatorname{conv}(V) \quad \text{for some } V \subseteq \{0,1\}^d.$$

For many purposes, we may assume that the 0/1-polytopes we consider are full-dimensional. Indeed, if a polytope is not full-dimensional, then it satisfies an equation $ax = \alpha$ with some $a_i \neq 0$, and then we may project by deleting the x_i-coordinate. Repeat as necessary.

Examples 3.45.
  (i) The d-cube $\operatorname{conv}(\{0,1\}^d) = [0,1]^d$ is a 0/1-polytope: dimension d, 2^d facets, $2d$ vertices.
  (ii) The d-simplex $\Delta_d = \operatorname{conv}(\{0, e_1, \ldots, e_d\})$ is a 0/1-polytope: dimension d, $d+1$ facets, $d+1$ vertices.
  (iii) The convex hull $\operatorname{conv}(\{e_1, \ldots, e_d, 1 - e_1, \ldots, 1 - e_d\})$ is a (non-regular) d-dimensional cross polytope: dimension d, $2d$ vertices, 2^d facets.
  (iv) The *Birkhoff polytope* B_n is the set of all matrices of size $n \times n$ with non-negative entries and row and column sums 1. For $n \geq 2$ has dimension $d = (n-1)^2$, d^2 facets given by $x_{i,j} \geq 0$, and $d!$ vertices, the permutation matrices. (There was an exercise on this.)
  (v) The *correlation polytope* is $\operatorname{COR}(n) = \operatorname{conv}\{xx^T : x \in \{0,1\}^n\}$. It is very interesting (and complicated), with *huge* numbers of facets (though there is no formal proof for large n).

Lemma 3.46. *The volume of any d-dimensional 0/1-polytope is an integer multiple of $\frac{1}{d!}$*

Proof. Any d-dimensional polytope can be *triangulated* without new vertices. (This will is not hard to do recursively, but will be discussed in more detail later.) Thus we have to discuss

only the case of a simplex. By symmetry, we may assume that 0 is a vertex, so the simplex is $\mathrm{conv}(\{0, a_1, \ldots, a_d\})$. Its volume is $\frac{1}{d} \det(a_1, \ldots, a_d)$, where $A = (a_1, \ldots, a_d)$ is an integer matrix (indeed, a 0/1-matrix). The determinant of an integer matrix is an integer. $\qquad\square$

Lemma 3.47 (Hadamard bound for 0/1-matrices). *The determinant of a 0/1-matrix $A \in \{0,1\}^d$ is bounded by*

$$|\det(A)| \leq \frac{(d+1)^{(d+1)/2}}{2^d}$$

Equality occurs whenever the simplex $\mathrm{conv}(\{0, a_1, \ldots, a_d\})$ is regular, and there is an $(d+1) \times (d+1)$ Hadamard matrix.

Proof. From A, we build a matrix

$$A' = \begin{pmatrix} 1 & 1^T \\ 0 & 2A \end{pmatrix}$$

of size $(d+1) \times (d+1)$ with $\det(A') = 2^d \det(A)$. Now subtract the first row from all others: This yields $A'' \in \{-1, 1\}^{(d+1) \times (d+1)}$ with $\det(A'') = \det(A)$.

Now apply the Hadamard bound: The columns of this matrix all have length $\sqrt{d+1}$, so $\det(A'') \leq (d+1)^{(d+1)/2}$. $\qquad\square$

By definition, a d-dimensional 0/1-polytope has at most 2^d vertices.

Lemma 3.48 (Bárány's upper bound, easy version). *A d-dimensional 0/1-polytope has not more than $d! + 2d$ facets.*

Proof. Indeed, $f_{d-1} \leq 2d + d!(1 - \mathrm{vol}_d(P))$.

The claim is true for the d-cube C_d. For any other d-dimensional 0/1-polytope $\mathrm{conv}(V)$, add a 0/1-vertex, to get $P' = \mathrm{conv}(V \cup v_0)$. For any facet that is destroyed by going from P to P', a pyramid of volume at least $\frac{1}{d!}$ is added. Thus

$$\mathrm{vol}_d(P') - \mathrm{vol}_d(P) \geq \frac{1}{d!}(f_{d-1}(P) - f_{d-1}(P')).$$

Now use induction on the number of missing vertices, or compare the telescope sums for $f_{d-1}(P) - f_{d-1}(C_d)$ and for $\mathrm{vol}_d(C_d) - \mathrm{vol}_d(P)$. $\qquad\square$

Theorem 3.49 (Bárány–Pór [2], Gatzouras, Giannopoulos & Markoulakis [10]). *A d-dimensional 0/1-polytope for large n can have a superexponential number of facets: For some $c > 0$ there are d-dimensional 0/1-polytopes P_d with*

$$f_{d-1} \geq \left(\frac{cd}{\log^2 d}\right)^{d/2}.$$

Theorem 3.50 (Alon & Vũ [1]; see also [26]). *Further remarkable facts:*

(i) *Large coefficients: The largest coefficient in the facet-description of a d-dimensional 0/1-polytope satisfies*

$$\frac{(d-1)^{(d-1)/2}}{2^{2d+o(d)}} \leq c(d) \leq \frac{d^{d/2}}{2^{d-1}}.$$

(ii) *Flat simplices: The smallest distance of a vertex from a facet it does not lie on in a d-dimensional 0/1-polytope satisfies*

$$\frac{2^{d-1}}{\sqrt{d}^{d+1}} \leq s(d) \leq \frac{2^{2d}}{\sqrt{d}^{d}} 2^{o(d)}.$$

For example ("large coefficients"),there is a 8-dimensional 0/1-simplex with a facet given by

$$12x_1 + 18x_2 + 3x_3 + x_4 - 10x_5 + 11x_6 - 4x_7 + 5x_8 \leq 19.$$

Exercise 3.51 (The "cube slice polytopes"). For $1 \leq k \leq d$ let

$$\Delta_{d-1}(k) := \{x \in [0,1]^d : x_1 + \cdots + x_d = k\}$$
$$= \mathrm{conv}\{x \in \{0,1\}^d : x_1 + \cdots + x_d = k\}.$$

(i) Show that $\Delta_{d-1}(k)$ is affinely equivalent to

$$\Delta'_{d-1}(k) := \{x \in [0,1]^{d-1} : k-1 \leq x_1 + \cdots + x_{d-1} \leq k\}$$
$$= \mathrm{conv}\{x \in \{0,1\}^{d-1} : k-1 \leq x_1 + \cdots + x_{d-1} \leq k\}.$$

(ii) Describe $\Delta_3(2)$.
(iii) Study how the hyperplane $H_k = \{x \in \mathbb{R}^d : x_1 + \cdots + x_d = k\}$ cuts the faces of the d-cube $[0,1]^d$. How do the resulting faces look like? Conversely, describe how the faces of $\Delta_{d-1}(k)$ arise from faces of $[0,1]^d$. (Hint: Distinguish vertices and higher dimensional faces!)
(iv) Show that the $\mathcal{H}$-description and the $\mathcal{V}$-description in the definition of $\Delta_{d-1}(k)$ give the same $(d-1)$-polytope.
(v) Show that $\Delta_{d-1}(k)$ and $\Delta_{d-1}(d-k)$ are combinatorially equivalent.
(vi) Show that for even d, $\Delta_{d-1}(\frac{d}{2})$ is *centrally symmetric*, i.e. there is a center point c such that for all $x \in \mathbb{R}^d$, $c + x \in \Delta_{d-1}(\frac{d}{2})$ if and only if $c - x \in \Delta_{d-1}(\frac{d}{2})$.
(vii) Show that $\Delta_{d-1}(1)$ and $\Delta_{d-1}(d-1)$ are simplices.
(viii) Show that for $1 < k < d-1$, $\Delta_{d-1}(k)$ has $2d$ facets. What are their combinatorial types? In particular, there are two different combinatorial types, except in the case $k = \frac{d}{2}$.
(ix) Show that $\Delta_{d-1}(k)$ is 2-simplicial and $(d-2)$-simple.
(x) Describe $\Delta_4(2)$: compute the f-vector, describe the facets.
(xi)* Show that the f-vector of $\Delta_{d-1}(k)$ is given by

$$f_{i-1}(\Delta_{d-1}(k)) = \left|\{[d] = A \uplus B \uplus C : |A| < k, |B| < d-k, |C| = i\}\right|$$
$$= \sum_{\substack{0 \leq s < k \\ k < s+i \leq d}} \binom{d}{s}\binom{d-s}{i}$$
$$= \sum_{\max\{-1,k-i\} < s < \min\{k,d-i+1\}} \frac{d!}{s!i!(d-s-i)!}$$

for $i > 1$. How about f_0?
(Hint: Every i-face of $[0,1]^d$ can be described in the form

$$\{x \in \mathbb{R}^d : x_j = 1 \text{ for } j \in A, x_j = 0 \text{ for } j \in A, 0 \leq x_j \leq 1 \text{ for } j \in C\},$$

for suitable sets A, B, C satisfying $A \uplus B \uplus C = [d]$ and $|C| = i$.)
(xii) Compute and plot the f-vector of $\Delta_{41}(21)$.

3.2 Three-dimensional polytopes

3.2.1 The graph

Definition 3.52 (The graph of a polytope)**.** The *graph* $G(P)$ of a polytope P is the "abstract" graph which has a vertex ("node") for each vertex ("0-face") of the polytope and an edge ("arc") for each edge ("1-face") of the polytope.

What can we say about the graphs of polytopes? Of 3-polytopes? Here are a first few observations:

- The graph of any d-polytopes is finite, and by definition it is a simple graph (i.e., it has no loops of parallel edges).

- It has at least $d + 1$ vertices [This condition is often forgotten!]

- Every vertex of the graph has degree at least d.

- The graph is connected. (This is not quite obvious, will be proved below.)

Proposition 3.53. *The graphs of* 3*-polytopes are planar.*

Proof 1. Find an interior point p_1 and from this point, project to a sphere centered at p_1. Thus the graph has been drawn on a sphere. □

Proof 2. Find a point p_0 that is beyond some facet F but beneath all other facets. From this point, what do you *see*? Equivalently, from this point project to the facet F. □

3.2.2 The graph is 3-connected

The graphs of 3-dimensional polytopes are (vertex) 3-connected.

Definition 3.54. A finite graph is k-*connected* (a.k.a. *vertex k-connected*) if it has at least $k + 1$ vertices, and any subgraph obtained by deleting at most $k - 1$ vertices is connected.

Remark 3.55. Menger's theorem states that a graph is k-connected if between any two (non-adjacent) vertices there are k paths that are disjoint except for the endpoints.

There is a similar statement for *edge*-connected graphs, where the paths would have no edge in common. This notion is not relevant for us here.

Theorem 3.56 (Balinski's Theorem)**.** *The graphs of d-dimensional polytopes are (vertex) d-connected.*

Proof. Induction on dimension, the cases $d \leq 2$ are clear.

Let S be a subset of the vertex set of P, with $|S| \leq d - 1$, so $\dim \mathrm{aff}(S) \leq d - 2$. Let v_0 be any vertex not in S (that exists), and let H be a hyperplane that contains $S \cup \{v_0\}$ (that exists).

After a coordinate transformation we may assume that $H = \{x \in \mathbb{R}^d : x_d = 0\}$, which allows us to talk about "above" and "below" the hyperplane (referring to points with $x_d > 0$ resp. $x_d < d$).

Case 1: If there are points in H both above and below H, then let F_+ and F_- be the faces of all points in P that have the maximal resp. minimal x_0. The graphs of F_+ and F_- are connected.

Now by Lemma 3.57, from every vertex of P that is above H, or that is on H but not in S (which in particular includes v_0), there is an edge upwards, and thus a monotonely-increasing path in the graph to a vertex in F_+ that does not meet S. Similarly, from every vertex of P that is below H, or that is on H but not in S (which in particular includes v_0), there is an edge downwards, and thus a monotonely-decreasing path in the graph to a vertex in F_- that does not meet S.

Putting all the paths together, we see that $G(P) \setminus S$ is connected.

Case 2: If P has only points above H but not below H, then we are already done with the first half of the argument of Case 1. Similarly if P has only points below H but not above H, then we only need the second half of the argument of Case 1. $\qquad\square$

Lemma 3.57. *Assume that v is a vertex of a convex polytope $P \subset \mathbb{R}^d$, which is not the highest vertex – that is, there is a vertex w that is higher, i.e., whose last coordinate has a higher value, $w_d > v_d$. Then there is some edge starting at v whose second endpoint is higher than v, that is, an edge $[v, w']$ with $w'_d > v_d$.*

Proof. The cone with apex v spanned by the edges of P starting at v contains the whole polytope: This was established when we discussed that the vertices of the vertex figure P/v correspond exactly to the edges of P at v.

Thus if there is no increasing edge at v, then v is maximal. $\qquad\square$

3.3   The Euler formula and some consequences

Lemma 3.58 (The Euler formula). *If any graph with n vertices, e edges, and c connected components is drawn in the plane, forming m faces (connected components of the complement: one of them unbounded, all the others bounded), then*

$$n - e + m = 1 + c.$$

In particular, every 3-polytope satisfies

$$f_0 - f_1 + f_2 = 2.$$

Proof. For example by induction on the number of edges, by plainly deleting edges one-by-one: For a graph without edges, we have $c = n$ and $m = 1$.

Adding an edge, either creates a new face (if the endpoints of the edge were already connected), or it reduces the number of connected components (otherwise), but not both.

See Eppstein [8] for twenty (!) proofs. $\qquad\square$

Corollary 3.59 (The upper bound theorem for 3-polytopes). *A 3-polytope with f_0 vertices has at most $3f_0 - 6$ edges and at most $2f_0 - 4$ facets.*

Proof. Every facet has at least three sides (edges), every edge is on exactly two facets, thus $3f_2 \leq 2f_1$, with equality if and only if the polytope is simplicial. From this we get ... $\qquad\square$

Corollary 3.60. *Every 3-polytope has triangles, quadrilaterals, and/or pentagons as faces. Indeed, if m_k denotes the number 2-faces with k vertices, then every 3-polytope satisfies*

$$3m_3 + 2m_4 + m_5 \geq 12$$

with equality if the polytope is simple and does have no faces with more than six sides.

Proof.

$$f_2 = m_3 + m_4 + m_5 + \dots$$

where

$$2f_1 = 3m_3 + 4m_4 + 5m_5 + 6m_6 + \dots$$

while

$$3f_0 \leq 2f_1.$$

Also

$$12 = 6f_0 - 6f_1 + 6f_2 \leq -2f_1 + 6f_2 = 3m_3 + 2m_4 + m_5 - m_7 - 2m_8 - \dots$$

$\square$

Remark 3.61. The regular polytopes in $\mathbb{R}^3$ can be classified using Euler's equation.

Exercise 3.62 (The f-vectors of 3-polytopes; Steinitz 1906 [22])**.** The f-vectors of 3-polytopes are given by the set

$$\{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 = 2, \quad f_2 \leq 2f_0 - 4, \quad f_0 \leq 2f_2 - 4.\}$$

3.4   Steinitz' theorem and three proofs

Theorem 3.63 ("Steinitz' theorem", 1910/1922 [23, 24])**.** *There ist a bijection between the isomorphism types of 3-connected planar finite graphs and the combinatorial types of 3-dimensional polytopes, induced by the obvious map $P \mapsto G(P)$.*

Proof.
"$\longleftarrow$" The graph is 3-connected: by Balinski's Theorem 3.56
    The graph is planar: projection into one face or onto the unit sphere.
    The map is one-to-one by Lemma 3.64 below.
"$\longrightarrow$" This direction requires that we demonstrate that/how for any 3-connected planar graph one can get a realization as the graph of a convex 3-polytope. This is non-trivial. In class/below we provide sketches for three different types of proofs.

$\square$

Lemma 3.64. *The combinatorial type of a 3-polytope is determined by its graph.*

Proof. There is a bijection between the vertices of a polytope and the vertices of its graph. The same holds for the edges. Each 2-face of the polyope can be associated to a non-seperating simple induced cycle in the graph (and vice versa). $\square$

Remark 3.65. This is not true for 4-dimensional polytopes! Consider the following example: A bipyramid over a tetrahedron ($\Delta_3 \oplus \Delta_1$) and a pyramid over $\Delta_2 \oplus \Delta_1$ have the same graph (K_6 minus an edge). The first polytope is simplicial while the second is not, hence they are not of the same type.

3.4.1 Steinitz type proofs

$G = G_1 \to G_2 \to \cdots \to G_n = K_4$, a sequence of 3-connected planar graphs obtained by small local operations, translate into a sequence of 3-polytopes $P = P_1 \leftarrow \cdots \leftarrow P_n = \langle\!\!\!\diamondsuit\!\!\!\rangle$.

Lemma 3.66. *Every planar* 3*-connected graph* G *has a sequence* $G \to G_1 \to \cdots \to G_n = K_4$, *where each operation* $G_i \to G_{i+1}$ *is either duality or a* Δ*-Y-transformation (perhaps better called a* ∇*-Y-transformation?). This operation replaces a triangle by the star graph* $K_{1,3}$ *(a tree with one internal vertex and three leaves) and then eliminates vertices of degree* 2 *that might have been created in this process (by replacing the two edges incident to such a vertex by a single edge).*

Corollary 3.67. *Every combinatorial type of a* 3*-polytope can be built from a tetrahedron by repeatedly taking the polar or cutting off a vertex of degree* 3*, which realizes a Y-∇-transformation.*

Corollary 3.68 (Barnette & Grünbaum 1970)**.** *For a* 3*-polytope, one can prescribe the shape of a face.*

Corollary 3.69 (Steinitz)**.** *Every planar* 3*-connected graph can be realized as a polytope with integer coordinates.*

For details of such a proof, see [25, Lecture 4].

3.4.2 Tutte's rubber band embeddings

Let G be a planar 3-connected graph. We may assume that G has a triangle face.

Lemma 3.70. *If G is* 3*-connected and planar, then either* G *or* G^* *has a triangle face.*

Proof. If not, then every face of G has ≥ 4 edges and every vertex has degree ≥ 4. By Euler's formula, this cannot happen. $\qquad\square$

Now fix the vertices of the triangle face in the plane, and for all other vertices take the coordinates that minimize the following energy function

$$E : \sum_{ij \in E} \tfrac{1}{2} \left((x_i - x_j)^2 + (y_i - y_j)^2 \right) = \min$$

$$\Leftrightarrow \sum_{ij \in E} (x_i - x_j)^2 = \min \ \text{ and } \ \sum_{ij \in E} (y_i - y_j)^2 = \min$$

which is interpreted as minimizing the energy of individual edges realized by (ideal) rubber bands.

This $E(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a quadratic function, so finding a minimum means solving a system of linear inequalities (which is unique, since G is connected). $\nabla E = 0$ yields a correct layout (i.e., with convex faces that don't overlap) of G in the plane which can be lifted to $\mathbb{R}^3$ to yield a convex 3-polytope.

Corollary 3.71 (Onn–Sturmfels, Richter-Gebert)**.** *If* G *is* 3*-connected, planar and has a triangle face, then it can be represented by a polytope with vertex coordinates in* $\{1, 2, \ldots, 43^n\}$*.*

For details of such a proof, see [18, Part IV].

3.4.3 Circle packing proofs

Theorem 3.72. *Every planar 3-connected graph G can be represented as a 3-polytope P which has all edges tangent to the unit sphere. This representation is unique up to orthogonal transformations if we require that the edge tangency points add to zero.*

Proof. Each polytope with those properties gives us two circle packings on the sphere which intersect orthogonally: the circle packing consisting of facet circles and the circle packing consisting of vertex horizon circles (1). This turns into a planar circle packing by stereographic projection (2). From this we can construct a planar graph (3) whose faces are quadrilaterals, called a quad graph, and by taking a subgraph, we obtain the desired planar, 3-connected graph (4). The reverse four-step-process yields a constructive proof of our statement. $\square$
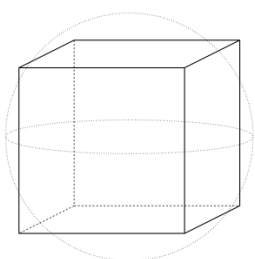
Figure 1: edge-tangent polytope
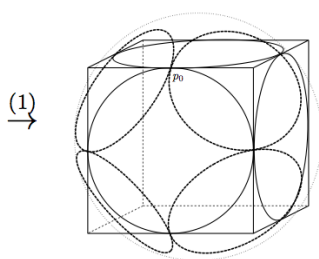
Figure 2: spherical circle packing
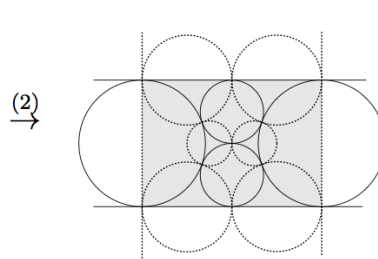
Figure 3: circle packing of a rectangle
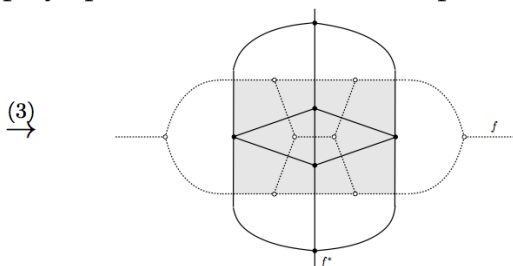
Figure 4: quad graph
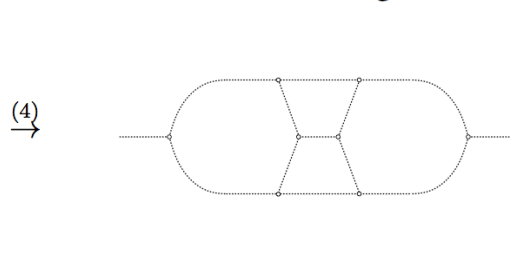
Figure 5: planar 3-connected graph

For details of such a proof, see [28, Lecture 1].

3.5 Three bits of history

-300: The Finale of the Elements: Euclid and the icosahedron
1498: Mistakes of a Genius: Leonardo and the Herrnhuther Stern
1525: A German Revolution: Dürer's geometry book from 1525, and his drawing of an ellipse
16??: Descartes finds Euler's Formula: from Descartes' lost manuscript, in Leibniz' copy
2011: Mae West: Geometry in Public – the sculpture at Effnerplatz in Munich
2013: Geometry in Public, II

Further reading: [29]. Merry Xmas!

3.6   Shellability, f-vectors, and the Euler–Poincaré equation

As discussed before the break, the Euler equation for 3-dimensional polytopes was "in essence" already discovered by Descartes in the 17th century.

The generalization of Euler's equation for d-dimensional polytopes was first stated in the middle of the 19th century by the Swiss mathematician Ludwig Schläfli, but his proof was incomplete, as it assumed that there is a "suitable" ordering on the facets of the polytope, which allows one to proceed by induction on the number of facets — in modern terms, Schläfli needed, but did not prove, that polytopes are "shellable." Schläfli's proof was completed when Peter Mani and his student Hans Bruggesser showed in 1969 that, indeed, all polytopes are shellable.

Long before this, the first complete proof of the Euler equation for d-polytopes was provided by Henri Poincaré around 1900, based on his new "Homology Theory."

Theorem 3.73 (The Euler–Poincaré formula). *For every convex d-dimensional polytope, the f-vector satisfies*

$$f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d.$$

Examples 3.74. For $d = 2$ we have $f_0 - f_1 = 0$: a polygon has as many sides as vertices; for $d = 3$ we have the classical Euler equation $f_0 - f_1 + f_2 = 2$; for $d = 4$ we obtain $f_0 - f_1 + f_2 - f_3 = 0$.

Definition 3.75 (Polytopal complex, pure, dimension, facets). A *polytopal complex* is a finite collection of polytopes in some $\mathbb{R}^N$ that contains all the faces of its polytopes, and such that the intersection of two polytopes in the collection is a face of both of them.

The *dimension* of a polytopal complex is the largest dimension of a polytope in the complex.

The polytopes in the complex are referred to as *faces* of the complex.

A complex is *pure* if all the inclusion-maximal faces, referred to as *facets*, have the same dimension.

Examples 3.76.
  (i) A drawing of a finite graph in the plane with straight edges and without crossings is a 1-dimensional complex (if the graph has at least one edge). It is pure if the graph has no isolated vertices.
 (ii) Any d-polytope (with all its faces) forms a pure polytopal complex $\mathcal{C}(P)$ of dimension d.
(iii) All the proper faces of a d-polytope form a pure polytopal complex of dimension $d - 1$, known as the *boundary complex* $\mathcal{C}(\partial P)$ of P.
(iv) If $P \subset \mathbb{R}^d$ is a convex d-polytope, then the faces of P that are "visible" from some point $x \notin P$ form a pure polytopal complex of dimension $d - 1$.

Definition 3.77 (Shelling/shellable). A k-dimensional pure polytopal complex $\mathcal{C}$ is *shellable* if either $k = 0$ (and thus the facets are vertices), or its facets have an ordering $F_1, F_2, \ldots, F_N$, called a *shelling* of $\mathcal{C}$ such that
  (i) the boundary complex of F_1 has a shelling, and

(ii) for $1 < i \leq N$, the intersection $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ is a non-empty union of $(k-1)$-dimensional faces of F_i,

$$F_i \cap (F_1 \cup \cdots \cup F_{i-1}) = G_1 \cup \cdots \cup G_\ell,$$

where $G_1, \ldots, G_\ell$ is a beginning of a shelling order for the boundary complex of F_i.

Examples 3.78.
- A shellable complex of dimension ≥ 1 is connected.
- A 1-dimensional complex (i.e., a graph) is shellable if and only if it is connected.
- A polytope complex $\mathcal{C}(P)$ is shellable if and only if its boundary complex $\mathcal{C}(\partial P)$ is shellable. In this case we say that the polytope is shellable.
- *Any* ordering of the facets of a d-simplex is a shelling order for the boundary complex.

The last point implies that for simplicial complexes condition (i) in Definition 3.77 is redundant, and condition (ii) simply says that $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ is a union of facets of F_i.

Exercise 3.79. How many shellings are there for a convex n-gon?

Exercise 3.80.
  (i) How many shellings are there for the (boundary of the) 3-dimensional cube?
 (ii) Show (using induction on dimension) that a facet ordering of the d-cube $F_1, \ldots, F_{2d}$ is *not* a shelling if and only if $F_1, \ldots, F_{2j}$ consists of j pairs of opposite facets, for some j, $1 \leq j < d$.
(iii)* Compute the number of shellings of the d-dimensional cube, for $1 \leq d \leq 10$.

Proposition 3.81. *For every polytope, the reversed order of a shelling is again a shelling order. That is, if $F_1, F_2, \ldots, F_N$ is a shelling order for $\mathcal{C}(\partial P)$, then so is $F_N, F_{N-1}, \ldots, F_1$.*

Proof. Induction on dimension d. Let for some $1 < i < N$

$$F_i \cap (F_1 \cup \cdots \cup F_{i-1}) = G_1 \cup \cdots \cup G_\ell,$$

where $G_1, G_2, \ldots, G_M$ is a shelling order for $\mathcal{C}(\partial F_i)$. Then we have

$$F_i \cap (F_{i+1} \cup \cdots \cup F_N) = G_{\ell+1} \cup \cdots \cup G_M,$$

since every facet of F_i is a facet of exactly one other F_j. But $G_M, G_{M-1}, \ldots, G_{\ell+1}$ is, by induction assumption, a beginning of a shelling order for $\mathcal{C}(\partial F_i)$. $\qquad\square$

Theorem 3.82 (Polytopes are shellable; Bruggesser & Mani [7])**.** *Every convex d-polytope is shellable.*

Proof. "Rocket flight" — see [25, proof of Thm. 8.11]. $\qquad\square$

Note that for a Bruggesser–Mani line shelling, the reversal is obtained by simply reversing the direction on the shelling line.

Remark 3.83. Theorem 3.82 implies that condition (i) in Definition 3.77 is redundant. Shelling can also be defined for cell complexes, then condition (i) becomes non-trivial.

The Euler-Poincaré characteristic can be defined for any polytopal complex. Theorem 3.73 says that $\chi(\mathcal{C}(\partial P)) = 1 - (-1)^d$. It can also be restated as

$$\chi(\mathcal{C}(P)) = 1$$

Indeed, $\chi(\mathcal{C}(P))$ differs from $\chi(\mathcal{C}(\partial P))$ by the summand $(-1)^d f_d = (-1)^d$.

Lemma 3.84. *Let $\mathcal{C}$ and $\mathcal{C}'$ be polytopal complexes such that $\mathcal{C} \cup \mathcal{C}'$ is also a polytopal complex. Then we have*

$$\chi(\mathcal{C}) + \chi(\mathcal{C}') = \chi(\mathcal{C} \cup \mathcal{C}') + \chi(\mathcal{C} \cap \mathcal{C}')$$

Proof. Every k-face of $\chi(\mathcal{C} \cup \mathcal{C}')$ is counted as many times at the left as at the right hand side: $(-1)^k$ times if it belongs to only one of the complexes, and $2 \cdot (-1)^k$ times if it belongs to both. $\square$

Remark 3.85. A *subcomplex* of a polytopal complex $\mathcal{C}$ is a subset of the set of faces of $\mathcal{C}$ closed under the operation of taking faces. The union or intersection of two subcomplexes is again a subcomplex. In particular, $F_1 \cup \cdots \cup F_{i-1}$ and $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ can be viewed as subcomplexes of $\mathcal{C}(P)$.

Proof of Theorem 3.73. Choose a shelling order $F_1, \ldots, F_N$ of the boundary complex of P (here $N = f_{d-1}$). We will prove

$$\chi(F_1 \cup \cdots \cup F_i) = \begin{cases} 1, & \text{for } 1 \le i < N \\ 1 - (-1)^d, & \text{for } i = N \end{cases}$$

by induction on d and i. Assume that we know this for $(d-1)$-dimensional polytopes, and look at what happens when we attach a facet F_i to $F_1 \cup \cdots \cup F_{i-1}$. Lemma 3.84 implies

$$\chi(F_1 \cup \cdots \cup F_i) = \chi(F_1 \cup \cdots \cup F_{i-1}) + \chi(F_i) - \chi(F_i \cap (F_1 \cup \cdots \cup F_{i-1}))$$
$$= 1 + 1 - \chi(G_1 \cup \cdots \cup G_\ell)$$

(Here the arguments of χ are understood as polytopal complexes, see Remark 3.85.)

If $i < N$, then $G_1 \cup \cdots \cup G_\ell$ is a shellable part (but not the whole) of $\mathcal{C}(\partial F_i)$, hence its Euler characteristic equals 1 by the induction assumption on d. (This is geometrically clear, and can be derived from Proposition 3.81.) For $i = N$ we have $G_1 \cup \cdots \cup G_\ell = \partial F_N$, which has Euler characteristic $1 - (-1)^{d-1}$ again by the induction assumption. In any case, this yields us the induction step on i. $\square$

Further reading: [25, Sections 8.1 "Shellable and non-shellable complexes" and 8.2 "Polytopes are shellable"]

3.7 Dehn–Sommerville, Upper Bound Theorem, and the g-Theorem

A *simplicial complex* is a polytopal complex, all of whose faces are simplices. For example, the boundary complex of a simplicial d-polytope is a pure $(d-1)$-dimensional simplicial complex.

Definition 3.86. Let $\mathcal{C}$ be a a pure $(d-1)$-dimensional simplicial complex, with f-vector $f(\mathcal{C}) = (1, f_0, \ldots, f_{d-1})$. Its h-vector $h(\mathcal{C}) = (1, h_1, \ldots, h_d)$ is given by

$$h_k \; := \; \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1},$$

that is,

$$h_k = f_{k-1} \; - \; (d-k+1)f_{k-2} \; + \; \binom{d-k+2}{2} f_{k-3} \; - \; \ldots$$
$$\ldots \; + \; (-1)^{k-1} f_0 \binom{d-1}{k-1} \; + \; (-1)^k \binom{d}{k}.$$

In particular, we have $h_0 = 1$, $h_1 = f_0 - d$, and

$$h_d \; = \; f_{d-1} \; - \; f_{d-2} \; + \; f_{d-3} \; - \; \ldots \; + \; (-1)^{d-1} f_0 \; + \; (-1)^d.$$

Also, it is easy to verify $h_0 + h_1 + \cdots + h_d = f_{d-1}$

Example 3.87. For the octahedron, we compute $f(P) = (1, 6, 12, 8)$ and $h(P) = (1, 3, 3, 1)$.

Remark 3.88. There is a nice trick, "Stanley's triangles," to compute the h-vector (see [25, Examples 8.20]).

Lemma 3.89. *The face numbers f_{k-1} are linear combinations of the h-numbers, with non-negative (!) integer coefficients:*

$$f_{k-1} \; = \; \sum_{i=0}^{k} \binom{d-i}{k-i} h_i$$
$$= \; h_k + (d-k+1)h_{k-1} + \cdots + \binom{d-1}{k-1} h_1 + \binom{d}{k} h_0.$$

Proof. We consider the f-*polynomial*

$$f(x) \; := \; f_{d-1} + f_{d-2}x + \cdots + f_0 x^{d-1} + f_{-1}x^d \; = \; \sum_{i=0}^{d} f_{i-1} x^{d-i}$$

and the h-*polynomial*

$$h(x) \; := \; h_d + h_{d-1}x + \cdots + h_1 x^{d-1} + h_0 x^d \; = \; \sum_{i=0}^{d} h_i x^{d-i}.$$

From the definition of the h-numbers we find $h(x) = f(x-1)$ and hence $f(x) = h(x+1)$. $\quad\square$

For pure simplicial complexes, the definition of shellability simplifies considerably. Condition (i) from Definition 3.77 is redundant, since the simplex is shellable (as is every polytope). Moreover, it can be shown that any ordering of the facets of a simplex is a shelling order. Hence condition (ii) simplifies to

(ii') For $1 < j \leq N$ the intersection of the facet F_j with the union of previous facets is a non-empty union of facets of F_j:

$$F_j \cap (F_1 \cup \cdots \cup F_{j-1}) = G_1 \cup \cdots \cup G_i$$

Let $\mathcal{C}$ be a shellable simplicial complex with a shelling order $F_1, \ldots, F_N$. We say that the facet F_j has *type i* (with respect to this shelling order) if its intersection with the union of previous facets consists of i facets of F_j. The facet F_1 has type 0.

Lemma 3.90. *The number of type i facets in a shelling of a pure simplicial complex equals h_i. In particular, it is independent of a choice of a shelling.*

Proof. Let F_j have type i. Count the faces that lie in $F_1 \cup \cdots \cup F_j$ but not in $F_1 \cup \cdots \cup F_{j-1}$. Let $G_1, \ldots, G_i$ be the "old" facets of F_j (those lying in $F_1 \cup \cdots \cup F_{j-1}$), and $G_{i+1}, \ldots, G_d$ be the new ones. Then the intersection

$$R_j := G_{i+1} \cap \cdots \cap G_d$$

is the unique minimal new face. It has i vertices and hence dimension $i - 1$. Every other new face H is characterized by $R_j \subseteq H \subseteq F_j$, so there are $\binom{d-i}{k-i}$ new faces of that have k vertices, i.e., dimension $k - 1$.

Thus every facet of type i adds $\binom{d-i}{k-i}$ to f_{k-1}, so that $f_{k-1} = \sum_{i=0}^{k} \binom{d-i}{k-i} t_i$ with t_i denoting the number of facets of type i. By comparing this with the formula in Lemma 3.89 we see that $t_i = h_i$. $\qquad \square$

Corollary 3.91. *The h-vector is non-negative for a shellable complex.*

Theorem 3.92 (The Dehn–Sommerville equations)**.** *For every simplicial d-polytope the h-vector is symmetric,*

$$h_k = h_{d-k} \qquad \text{for } 0 \leq k \leq d.$$

Proof. Reverse the shelling! $\qquad \square$

Note that $h_0 = h_d$ is the Euler-Poincaré formula (for simplicial polytopes).

Theorem 3.93 (The Upper Bound Theorem, McMullen [16])**.** *For any $n > d \geq 1$ and $0 \leq k \leq d$, the cyclic polytope $C_d(n)$ has the maximal number of k-faces among all d-polytopes with n vertices, that is,*

$$f_k(P) \leq f_k(C_d(n)).$$

Observe that the claim is quite trivial for $k \leq \frac{d}{2}$, as the cyclic polytopes are neighborly.

Proof. First, we may assume that P is simplicial, since "pulling the vertices (one after the other)" produces a simplicial polytope while (weakly) increasing the components of the f-vector.

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Then, a clever induction using carefully chosen Bruggesser–Mani shellings yields that

$$h_k(P) \leq h_k(C_d(n)) \qquad \text{for all } k,$$

which is more than enough, due to Lemma 3.89. During the proof we also find

$$h_k(C_d(n)) = \binom{n - d - 1 + k}{k} \qquad \text{for } k \leq \lfloor \tfrac{d}{2} \rfloor$$

See [16] resp. [25, Sect. 8.4]. □

Details of the proof:

Definition 3.94. For a simplicial complex $\mathcal{C}$ and a vertex v of $\mathcal{C}$, the *star* $\operatorname{star}(v, \mathcal{C})$ of v in $\mathcal{C}$ is the subcomplex consisting of all faces that contain v, and all their faces. The *link* $\operatorname{link}(v, \mathcal{C})$ of v in $\mathcal{C}$ consists of all $G \in \operatorname{star}(v, \mathcal{C})$ that do not contain v. Thus, the star of a vertex is a cone over its link.

Lemma 3.95. *Let $\mathcal{C}$ be a shellable simplicial complex. Then the restriction of any shelling order of $\mathcal{C}$ to $\operatorname{star}(v, \mathcal{C})$ is a shelling of $\operatorname{star}(v, \mathcal{C})$. It also induces a shelling order for $\operatorname{link}(v, \mathcal{C})$.*

Proof. After checking this for the star, note that the facets of the link are in a 1-1 correspondence with the facets of the star. □

Lemma 3.96. *For every polytope P and every vertex v of P there is a shelling order for $\mathcal{C}(\partial P)$, in which the facets of $\operatorname{star}(v, \mathcal{C}(\partial P))$ come first.*

Proof. There is a generic point outside P, from which only the star of v is visible. □

The inequality $h_k(C_d(n)) \leq \binom{n-d-1+k}{k}$ is proved by induction on k.

Lemma 3.97. *For every shellable pure $(d-1)$-dimensional simplicial complex $\mathcal{C}$ we have*

$$\sum_{v \in \operatorname{vert}(\mathcal{C})} h_k(\mathcal{C}/v) = (k+1)h_{k+1}(\mathcal{C}) + (d-k)h_k(\mathcal{C})$$

Here we denote by $\mathcal{C}/v$ the link of v in $\mathcal{C}$.

Lemma 3.98. *Let $\mathcal{C} = \mathcal{C}(\partial P)$ be the boundary complex of a simplicial polytope. Then we have*

$$h_k(\mathcal{C}/v) \leq h_k(\mathcal{C}) \qquad \text{for all } v \in \operatorname{vert}(\mathcal{C}),$$

and the equality holds for all $k \leq l$ iff P is $(l+1)$-neighborly.

Summing up the inequalities of Lemma 3.98 and combining the result with the equation of Lemma 3.97 yields

$$(k+1)h_{k+1} \leq (n-d+k)h_k,$$

which can be used as induction step for $h_k(C_d(n)) \leq \binom{n-d-1+k}{k}$.

Further reading: [25, Sections 8.3 "h-vectors and the Dehn–Sommerville equations" and 8.4 "The Upper bound theorem"].

3.8 Lower bound theorem, *g*-theorem, the set of all *f*-vectors

Theorem 3.99 (The Lower Bound Theorem; Barnette 1971/1973)**.** *Let P be a simplicial d-polytope with n vertices, then*

$$f_i(P) \geq f_i(\mathrm{Stack}_d(n)) = \begin{cases} \binom{d+1}{i+1} + (n-d-1)\binom{d}{i} & \text{for } i < d-1, \\ (d+1) + (n-d-1)(d-1) & \text{for } i = d-1, \end{cases}$$

with equality for all i only if $d = 3$ or if P is a stacked polytope.

Proof. This is elementary graph theory, using induction etc., see e.g. Brøndsted [6]. □

Remark: valid also for pseudomanifolds
Remark: not valid for non-simplicial polytopes!

Theorem 3.100 (The g-Theorem; conjectured by McMullen 1971; "sufficiency" of McMullen's conditions by Billera & Lee 1980 [3, 4]; "necessity" by Stanley 1980 [21])**.**
A sequence $(1, f_0, \ldots, f_{d-1})$ is the f-vector of a simplicial d-polytope $\iff$
$(1, g_1, \ldots, g_{\lfloor d/2 \rfloor}) := (h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ *is an M-sequence $\iff$*
$1, f_0, \ldots, f_{d-1} = gM_d$ *for an M-sequence g.*
Here

- *M_d is the* Björner matrix

$$M_d := \left(m_{jk} \right)_{jk} := \left(\binom{d+1-j}{d+1-k} - \binom{j}{d+1-k} \right)_{0 \leq j \leq \lfloor d/2 \rfloor, \ 0 \leq k \leq d} \in \mathbb{Z}^{(\lfloor d/2 \rfloor + 1) \times (d+1)}.$$

For example, we compute

$$M_1 = (1\ 2), \qquad\qquad M_2 = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 4 & 6 & 4 \\ 0 & 1 & 3 & 2 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 1 & 5 & 10 & 10 & 5 \\ 0 & 1 & 4 & 6 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}.$$

- *An M-sequence is the f-vector of a multicomplex/complex of monomials; it is characterized as follows (a proposition due to Macaulay): $(1, g_1, \ldots, g_{\lfloor d/2 \rfloor}) \in \mathbb{Z}^{\lfloor d/2 \rfloor + 1}$ is an M-sequence if and only*
  - *$g_i \geq 0$ for all i,*
  - *$g_{k-1} \geq \partial^k g_k$ for all k,*

  where the "boundary operators" ∂^k can be defined as follows: Using the notation $\left(\binom{n}{k} \right) := \binom{n+k-1}{k}$, we can write every number $m \geq 0$ uniquely as unique expansion of n of the form

$$m = \left(\binom{b_k}{k} \right) + \left(\binom{b_{k-1}}{k-1} \right) + \ldots + \left(\binom{b_2}{2} \right) + \left(\binom{b_1}{1} \right)$$
$$\text{with } b_k \geq b_{k-1} \geq \ldots \geq b_2 \geq b_1 \geq 0$$

and with using the b_i's from this expansion, we set

$$\partial^k(m+1) \ := \ \left(\!\!\binom{b_k}{k-1}\!\!\right) + \left(\!\!\binom{b_{k-1}}{k-2}\!\!\right) + \cdots + \left(\!\!\binom{b_2}{1}\!\!\right) + \left(\!\!\binom{b_1}{0}\!\!\right).$$

For example, $(1, g_1, g_2) \in \mathbb{Z}^3$ is an M-sequence iff the g_i are nonnegative an $g_1 \geq \binom{g_2}{2}$ — see Problem Sheet 11.

This remarkable theorem — certainly a highlight of modern polytope theory — gives us a complete combinatorial description of the f-vectors of all *simplicial* convex polytopes.

A similar description of the set

$$\mathcal{F}_d \ := \ \{f = (1, f_0, \ldots, f_{d-1} : f \text{ is the } f\text{-vector of a } d\text{-dimensional convex polytope }\}$$

is out of reach for larger d.

More precisely, $\mathcal{F}_1$ and $\mathcal{F}_2$ are trivial to work out, while $\mathcal{F}_3$ was determined by Steinitz in 1906 — see Exercise 3.62.

Lemma 3.101 (Grünbaum [12])**.** *The affine hull of the set $\mathcal{F}_d \subseteq \mathbb{R}^{d+1}$ has dimension $d-1$.*

Proof. Indeed, the fact that the first entry of each f-vector is 1 and the Euler–Poincaré equation together yield that $\dim(\mathrm{aff}(\mathcal{F}_d)) \leq d-1$. The "$\geq$" inequality is left as an exercise. $\qquad\square$

A good description of the 3-dimensional set $\mathcal{F}_4$ seems to be out of reach in the moment. Indeed, even an approximate description is not in sight. For example (and that seems to be a major obstacle) it is not at all clear whether the quotient

$$\Phi \ := \ \frac{f_1 + f_2}{f_0 + f_3}$$

known as *fatness* of a 4-polytope can be arbitrarily large. See [27].

However, some partial information is available. For example, Grünbaum also proved that

$$\{(f_0(P), f_3(P)) : P \text{ is a convex 4-polytope }\}$$
$$= \ \{(f_0, f_3) \in \mathbb{Z}^2 : f_3 \leq \tfrac{1}{2}f_0(f_0 - 3), \ f_0 \leq \tfrac{1}{2}f_3(f_3 - 3), \ f_0 + f_3 \geq 10\}$$

and called this "quite easy." See [12, p. 292]:

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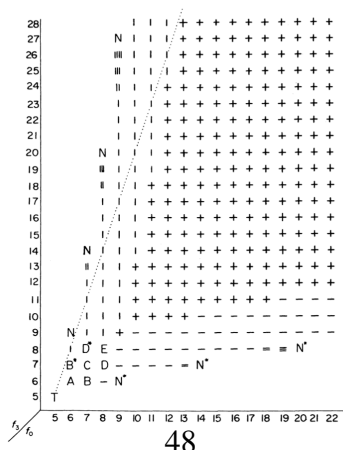
3.9 Graphs of d-polytopes

What do we know about the graphs of d-dimensional polytopes?

The graphs of 2-polytopes are the simple cycles, C_n, $n \geq 3$.

The graphs of 3-polytopes are the 3-connected planar graphs (Steinitz' Theorem 3.63).

Exercise 3.102. Describe the graphs of d-polytopes with $d + 1$ or $d + 2$ vertices.

We also know, for example, that each graph of a d-polytope contains a subdivision of K_{d+1}. Thus, in particular, they are not planar for $d \geq 4$.

The graphs of d-polytopes, $d \geq 4$, have not been characterized. It seems that a complete answer is completely out of reach. However, despite some negative results, there are also surprising positive statements.

3.9.1 The graph of a d-polytope is d-connected; moreover ...

Balinski's theorem 3.56 says that the graphs of d-polytopes are d-connected.

However, there are also more subtle connectivity properties, measured by a quantity called "degree of total separability":

Theorem 3.103 (Klee 1964, see [12, Sect. 11.4])**.** *By removing at most n vertices, the graph of a d-polytope can be decomposed into at most the following number of connected components:*

$$s(n,d) \leq \begin{cases} 1 & \text{for } n \leq d - 1, \\ 2 & \text{for } n = d, \\ f_{d-1}(C_d(n)) & \text{for } n > d. \end{cases}$$

Proof. For $n \leq d - 1$ this follows from Balinski's theorem, for $n = d$ it also follows from our proof method for Balinski's theorem.

For $n > d$ let P be the polytope we consider, and let $P' := \text{conv}(V')$ be the subpolytope given as the convex hull of the set V' of n vertices in the separating set. This P' has n vertices. If P' is not full-dimensional, then again the proof method for Balinski's theorem yields that removing its vertices from the graph of P results in at most 2 components. Thus we may assume that P' is full-dimensional, so by the upper bound theorem, we get that P' has not more than $f_{d-1}(C_d(n))$ facets.

Now all vertices in $V \setminus V'$ lie beyond some facet of P', and the subgraph of $G(P) \setminus V'$ on the vertices that lie beyond some particular facet of P' is connected (same argument as in the proof of Balinski's theorem). Thus $G(P) \setminus V'$ has at most $f_{d-1}(C_d(n))$ connected components. $\quad\square$

Examples 3.104. $G = K_6 * \overline{K_{10}}$ is not the graph of a d-polytope, for any d: As it is not planar, $d \leq 3$ is excluded. As it is not 7-connected (removing the $n = 6$ vertices of the K_6 decomposes the graph into 10 components), $d \geq 7$ is excluded. Similarly, for $d = 6$ the removal of the K_6 would be allowed to give only 2 components, for $d = 5$ only 6 components, and for $d = 4$, where $C_4(6) \cong \Delta_2 \oplus \Delta_2$, only at most $f_3(C_4(6)) = 9$ components.

Similarly, $G = K_6 * \overline{K_6}$ is dimensionally unique: If this is the graph of a d-polytope, then $d = 4$.

3.9.2 The graph does not determine the combinatorial type, but …

Recall the example: $\Delta_3 \oplus \Delta_1$ and a pyramid over $\Delta_2 \oplus \Delta_1$ have the same graph (K_6 minus an edge). Even two simplicial polytopes can have the same graph, without having the same combinatorial type.

Theorem 3.105 (Blind & Mani [5]; Kalai [13])**.** *The combinatorics of a simple polytope is determined by the graph.*

Proof (Sketch). Bruggesser–Mani line shelling orders on the facets of a simplicial polytope correspond to vertex orderings of the dual polytope, which is simple, by a linear functional.

Given the graph, which orderings are the *good* ones induced by linear orderings, or (more generally) the shelling orderings? Well, they are the ones that induce a *unique sink* on each non-empty face.

Thus one can look at the quantity $\sum_{v \in V} 2^{\delta_\mathcal{O}^+(v)}$, which counts all the non-empty faces — where the minimum is achieved for the shelling orderings, where each face is counted only once.

Now we characterize the graphs of facets: These are the $(d-1)$-connected $(d-1)$-regular subgraphs that are initial in some *good* vertex ordering. $\square$

Only recently, Eric Friedman has shown that the reconstruction can be done efficiently [9].

3.9.3 The graphs can have high diameters, but …

Lemma 3.106. *If P is a 3-polytope with n facets, then the diameter of the graph is not larger than $\lfloor \frac{2}{3}n \rfloor - 1$.*

Proof. The 3-polytope has at most $2n - 4$ vertices, so at most $2n - 6$ other than the two we are trying to connect. So one of the three vertex disjoint paths guaranteed by Balinski's theorem uses only at most $\lfloor \frac{1}{3}(2n-6) \rfloor$ of these vertices, so its length is at most $\lfloor \frac{1}{3}(2n-6) \rfloor + 1$. $\square$

Exercise 3.107. Show that Lemma 3.106 is sharp for all $n \geq 4$.

Conjecture 3.108 ("The Hirsch conjecture" – 1957)**.** *The maximal diameter of the graph of a simple d-polytopes with n facets, denoted $\Delta(d, n)$, satisfies*

$$\Delta(d, n) \leq n - d.$$

This conjecture is certainly plausible. It was proved in many cases, and also many equivalent statements. For example, it is *equivalent* to the claim that between any two vertices of a simple polytope there is a path that does not "revisit" a facet.

Nevertheless, the Hirsch conjecture is **false** in high dimensions:

Theorem 3.109 (F. Santos in 2011 [19])**.** *There is a 43-dimensional simple polytope with 86 facets of diameter at least 44.*

Santos' work has been extended: In particular, there are counter-examples for $d \geq 20$, according to Matschke et al. [15].

Nevertheless, we know little about diameters of polytopes: For example, is it true that $\Delta(n, d)$ is bounded by a polynomial in n and d? Is $\Delta(n, d) \leq d(n - d)$?

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4 Combinatorial geometry

4.1 Arrangements of points and lines

Definition 4.1. A *(planar) point configuration* is a finite subset $S \subset \mathbb{R}^2$ of the Euclidean plane.

Proposition 4.2 (Sylvester–Gallai 1893/1944)**.** *Every finite set of n points in the plane, not all on a line, n large, defines an "ordinary" line, which contain exactly 2 of the points.*

This is one of the "highlights" at the start of this course. The problem was stated by Sylvester in 1983, a solution by Tibor Gallai [Grünwald] was published by Erdős in 1944. The "BOOK proof" is due to L. M. Kelly [1]: It considers the smallest distance between a point p_0 in the set S and a line ℓ_0 spanned by points in S, and then shows that the line ℓ is an "ordinary" line, which contains exactly 2 of the points.
The definitive result is very recent, from 2013:

Theorem/Problem 4.3 (Green–Tao 2012 [3])**.** *There is a number n_0 such that for any finite set of $n \geq n_0$ points in the plane, not all on a line,*
if n is even, then there are at least $n/2$ "ordinary" lines,
if n is odd, then there are at least $3\lfloor \frac{1}{4}n \rfloor$ ordinary lines.

Remark 4.4. A finite set of points in the plane is "in general position" if no three of the points lie on a line. However, we might impose stronger conditions, such as that no two points lie on a vertical line (see below), etc. Thus, "in general position" is redefined as needed.

Theorem 4.5 (Erdős–Szekeres, 1935)**.** *For any $k \geq 3$ there is a smallest number $n(k)$ such that any set of $n(k)$ points in the plane in general position contains a subset of k points in convex position.*

We sketch two proofs of this theorem, the first via Ramsey numbers, the second one by induction using cups and caps.

Lemma 4.6. $n(3) = 3$; $n(4) = 5$.

Proof. The first statement is trivial, for the second one we may assume that the convex hull of the 5 points we consider forms a triangle (otherwise we are done). Then the two points in the interior together with two vertices of the triangle do it. $\square$

Theorem 4.7 (Ramsey)**.** *For any $k \geq p \geq 1$ and $r \geq 1$ there is a smallest number $n = n(k, p, r)$ sucht that if we color the p-subsets of an n-set X with r colors, then the p-subsets of some k-subset $Y \subseteq X$ all get the same color.*

Note: The Ramsey numbers $n(k, p, r)$ are typically huge, and *very* hard to compute or even estimate. However: $n(3, 2, 2) = 6$ is the classical statement that among any six people at a party, three know each other, or three don't know each other. That is, if you blue/red color the edges of K_6, then there is a monochromatic triangle. For K_5 this is wrong.

Proof 1 of the Erdős–Szekeres theorem. We 2-color the 4-subsets of X by "convex" and "non-convex." If $|X| \geq n(k, 4, 2)$, then by the Ramsey theorem 4.7 either there is a k-subset of $Y \subseteq X$ all of whose 4-subsets are "convex," but this implies that Y is in convex position. Or there is a k-subset of $Y \subseteq X$ all of whose 4-subsets are "non-convex," but this is impossible for $k \geq 5$ by Lemma 4.6. $\qquad\square$

Proof 1 of the Erdős–Szekeres theorem. Assume X is in general position, which in this proof means that no three points lie on a line, and no two span a vertical line (have the same x-coordinate).

We define a k-*cup* and an ℓ-*cap* as a sequence of k resp. ℓ points from X sorted by increasing x-coordinate, such that the subsequent slopes increase resp. decrease.

Let $f(k, \ell)$ be the smallest n such that any n-set in $\mathbb{R}^2$ in general position contains a k-cup or an ℓ-cap.

To prove the theorem it suffices to establish that $f(k, \ell)$ is finite, as all caps and all cups are in convex position, so $n(k) \leq f(k, \ell)$.

Claim: $f(k, \ell) \leq \binom{k+\ell-4}{k-2} + 1$.

This is clearly true if $k \leq 2$ or $\ell \leq 2$. Thus we can proceed by induction, assuming that $k \geq 3$ and $\ell \geq 3$, and establishing the following:

Claim: $f(k, \ell) \leq f(k-1, \ell) + f(k, \ell-1) - 1$.

Now assume that X is a set of size $f(k-1, \ell) + f(k, \ell-1) - 1$ without an ℓ-cap.

Let $E \subseteq X$ be the set of points that are not the right endpoints of $(k-1)$-caps.

Then $X \setminus E$ does not contain a $(k-1)$-cap, so $|X \setminus E| \leq f(k-1, \ell) - 1$, so $|E| \geq f(k, \ell-1)$.

So either E contains a k-cup, then we are done, or it contains an $(\ell-1)$-cap.

Thus we get a configuration that joins a $(k-1)$-cup whose right endpoint is the left endpoint of an $(\ell-1)$-cap (of points in E). But whenever we in this way connect a $(k-1)$-cup with an $(\ell-1)$-cap, we either get a ℓ-cap, which would contradict our assumption, or we get a k-cap. So we are done. $\qquad\square$

Die Abschätzung über "cups" und "caps" aus diesem Bereich ist sogar scharf, aber die Schranke, die man für $n(k)$ bekommt, trotzdem nicht. Da wissen wir im Moment

$$2^{k-2} \leq n(k) \leq \binom{2k-5}{k-2} + 2 < 4^k,$$

... zumindest ist das der Stand von Matoušek 2002, auf dessen Darstellung wir uns hier gestützt haben [4].

4.2   Line arrangements, hyperplane arrangements, and zonotopes

Definition 4.8 (Line arrangement)**.** A *line arrangement* $\mathcal{A}$ is a set of $n \geq 0$ distinct lines in $\mathbb{R}^2$; the connected components of its complement $\mathcal{A} \setminus \bigcup \mathcal{A}$ are called *regions* (or *chambers*). Some of the regions are unbounded, some may be bounded.

The arrangement is called *essential* if two of the lines intersect, that is, some point in the plane is an intersection of lines.

An arrangement is *central* if all lines are linear subspaces, that is, contain the origin.

Thus a line arrangement is essential and central if and only if $\bigcap \mathcal{A} = \{0\}$.

Proposition 4.9 (Buck's theorem). *An arrangement of n lines has at most $\binom{n+1}{2} + 1$ regions, with equality if and only if no two lines are parallel and no three of them intersect in a point.*

Proof. This is true for $n = 0, 1$; use induction. $\quad\square$

There is a "naive" way to set up duality, where we map each line $a^t x = b$ to the point $\frac{1}{b}a$, and the lines through the origin are mapped to the points at infinity, etc. This is avoided by the following nicer and often more useful version.

Proposition 4.10 (Duality). *There is a bijection between*

$$\{\text{points in } \mathbb{R}^2\} \quad \longleftrightarrow \quad \{\text{non vertical lines in } \mathbb{R}^2\}$$
$$(p_1, p_2) \quad \overset{*}{\longleftrightarrow} \quad \{(x, y) \in \mathbb{R}^2 : y = p_1 x - p_2\}.$$

Under this duality, a point on a non-vertical line $p \in \ell$
is mapped to a non-vertical line though a point, $p^* \ni \ell^*$.
Moreover, if the point p lies above the non-vertical line ℓ,
then the point ℓ^* lies above the non-vertical line p^*.

Proof. Let the point be (p_1, p_2) and the line $\{(x, y) : y = m_1 x + m_2$, and compute:

$$p \in \ell \iff p_2 = m_1 p_1 + m_2$$
$$\iff -m_2 = p_1 m_1 - p_2$$
$$\iff (m_1, -m_2) \in \{(x, y) \in \mathbb{R}^2 : y = p_1 x - p_2\} \iff p^* \ni \ell^*.$$

and similarly for

$$p \text{ above } \ell \iff p_2 > m_1 p_1 + m_2$$
$$\iff -m_2 > p_1 m_1 - p_2$$
$$\iff (m_1, -m_2) \in \{(x, y) \in \mathbb{R}^2 : y > p_1 x - p_2\} \iff \ell^* \text{ above } p^*.$$

$\quad\square$

Exercise 4.11. Dualize Sylvester–Gallai and Erdős–Szekeres theorems.

Exercise 4.12. Generalize Proposition 4.10 to $\mathbb{R}^d$

Definition 4.13 (Hyperplane arrangements). A *hyperplane arrangement* $\mathcal{A}$ is a finite set of $n \geq 0$ distinct affine hyperplanes in $\mathbb{R}^d$, $d \geq 1$. The connected components of its complement $\mathcal{A} \setminus \bigcup \mathcal{A}$ are called *regions* (or *chambers*). Some of the regions are unbounded, some may be bounded.

The *faces* of the hyperplane arrangement are the faces of (the closures of its) regions, which are polyhedra.

The hyperplane arrangement is called *essential* if some d of the hyperplanes intersect in a single point, that is, some point in $\mathbb{R}^d$ is an intersection of hyperplanes of the arrangement.

A hyperplane arrangement is *central* if all lines are linear subspaces, that is, contain the origin.

Lemma 4.14. *If we label and orient the hyperplane arrangement, that is, number its hyperplanes $H_1, \ldots, H_n$ and choose a "positive side" H_i^+ for each of them, then this determines a labelling of the non-empty faces of the hyperplane arrangement by vectors in $\{+, 0, -\}^n$.*

Definition 4.15. For any arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{R}^d$, there is an *associated central arrangement* $\widehat{\mathcal{A}}$ of $n+1$ hyperplanes in $\mathbb{R}^{d+1}$ obtained as follows. Associate to each hyperplane $H_i = \{x \in \mathbb{R}^d : a_i^t x = \beta_i\}$ the linear hyperplane $\widehat{H}_i := \{x \in \mathbb{R}^{d+1} : a_i^t x = \beta_i x_{d+1}\}$ which can also be characterized by $H_i \times \{1\} \subset \widehat{H}_i$. Then add the extra hyperplane $\widehat{H}_n := \mathbb{R}^d \times \{0\} = \{(x, 0) : x \in \mathbb{R}^d\}$.

The central hyperplane arrangement $\widehat{\mathcal{A}}$ in $\mathbb{R}^{d+1}$ obtained this way has the number of regions of $\mathcal{A}$. If $\mathcal{A}$ is essential, then so is $\widehat{\mathcal{A}}$. Thus for many purposes this definition (with the extra new hyperplane) is "the right one."

Definition 4.16. Let $\mathcal{A}$ be an essential and central arrangement of n (labeled) hyperplanes in $\mathbb{R}^d$. The *intersection lattice* $L(\mathcal{A}) \subseteq 2^{\mathcal{A}}$ of $\mathcal{A}$ is the set of all intersections of subarrangements of $\mathcal{A}$, ordered by reversed (!) inclusion. Its maximal element $\hat{1} = \mathbb{R}^d$ it interpreted as the intersection of the empty subarrangement.

The *face lattice* $\mathcal{L}(\mathcal{A})$ is the partially ordered set of all labels $X(F) \in \{-1, 0, \}^n$ associated to nonempty faces of $\mathcal{A}$, partially ordered componentwise by "$0 < +$" and "$0 < -$", with an extra maximal element $\hat{1}$ added.

Remark 4.17. For any central arrangement $\mathcal{A}$, the poset $L(\mathcal{A})$ is a is a "geometric" (semimodular, atomic) lattice, thus defines a "matroid." A popular sport is to read properties (such as the number of chambers) from $L(A)$ alone. Not treated here.

Proposition 4.18. *The face lattice $\mathcal{L}(\mathcal{A})$ of any essential central hyperplane arrangement in $\mathbb{R}^d$ is the face lattice of a d-dimensional polytope $Z^*(\mathcal{A})$.*

Proof. Let the hyperplanes be $H_1, \ldots, H_n$, where $H_i = \{x \in \mathbb{R}^d : a_i^t x = 0\}$. Consider the function

$$f : \mathbb{R}^d \to \mathbb{R}, \qquad f(x) = |a_1^t x| + |a_2^t x| + \cdots + |a_1^t x|.$$

This function is strictly positive, except in the origin. It is piecewise-linear, convex, and its domains of linearity are exactly the regions of the arrangement. Thus

$$Z^*(\mathcal{A}) := \{x \in \mathbb{R}^d : |a_1^t x| + |a_2^t x| + \cdots + |a_1^t x| \leq 1\}$$

is a centrally-symmetric d-polytope that "spans" the arrangement. $\qquad \square$

Definition 4.19. A *zonotope* $Z \subset \mathbb{R}^d$ is a polytope of the form

$$
\begin{aligned}
Z(v_1, \ldots, v_n) &:= \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_1, \ldots, \lambda_n \in [-1, 1]\} \\
&= [-v_1, v_1] + \cdots + [-v_n, v_n]
\end{aligned}
$$

for a finite set of vectors $V = \{v_1, \ldots, v_n \in \mathbb{R}^d\}$, that is, a Minkowski sum of a collection of intervals.

We should assume here that none of the vectors v_i is a multiple of another one (minimal representation), then the vectors v_i are in bijection with the parallel classes of edges of $Z(v_1, \ldots, v_n)$, known as the *zones* of the zonotope.

By definition, a zonotope is the linear image (projection image) of the standard cube $[-1, +1]^n$ under the map $e_i \mapsto v_i$.

One can show that a polytope is a zonotope if the polytope and all its faces are centrally symmetric.

Examples include the centrally-symmetric $2n$-gons, and the standard n-cube $[-1, +1]^n$.

Proposition 4.20. *The polytope $Z^*(\mathcal{A})$ constructed in the proof of Theorem 4.18 is the dual of the zonotope generated by the unit normal vectors of the hyperplanes of $\mathcal{A}$.*

Proof.

$$
\begin{aligned}
Z^*(\mathcal{A}) &= \{x \in \mathbb{R}^d : |a_1^t x| + \cdots + |a_1^t x| \leq 1\} \\
&= \{x \in \mathbb{R}^d : \pm a_1^t x \pm \cdots \pm a_1^t x \leq 1 \text{ for all signs }\} \\
&= \{x \in \mathbb{R}^d : \lambda_1 a_1^t x + \cdots + \lambda_n a_n^t x \leq 1 \text{ for arbitrary } \lambda_i \in [-1, 1]\} \\
&= \{x \in \mathbb{R}^d : (\lambda_1 a_1 + \cdots + \lambda_n a_n)^t x \leq 1 \text{ for arbitrary } \lambda_i \in [-1, 1]\} \\
&= Z(a_1, \ldots, a_n)^*.
\end{aligned}
$$

$\square$

Theorem 4.21 (Shannon's theorem [5] [2, pp. 49-50])**.** *Every central arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{R}^{d+1}$ has at least $2n$ simplicial regions.*
More precisely, each hyperplane H_i has at least $2(d+1)$ simplicial regions with a facet in H_i.

Proof. Induction on dimension, looking at the affine arrangement that we get by dehomogenizing by an additional "generic hyperplane." This yields an affine d-dimensional arrangement, in which each hyperplane is adjacent to a bounded simplicial region, which we again get from the Sylvester–Gallai–Kelly idea, that is, by looking at a vertex that has smallest distance to the hyperplane we look at. $\square$
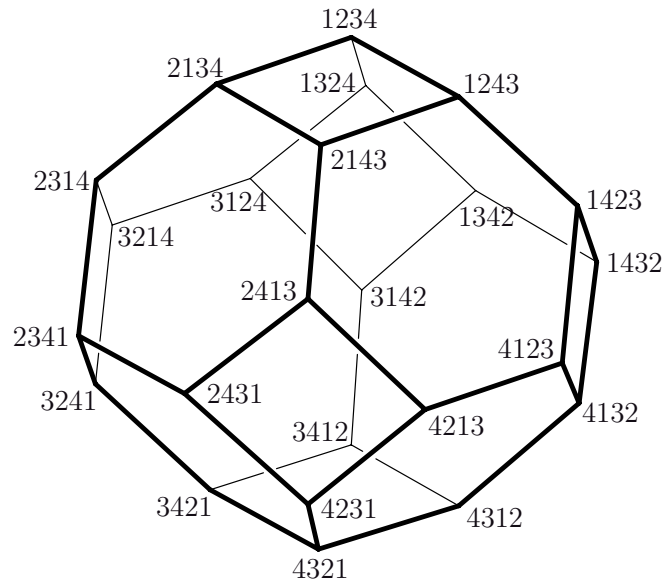

Definition 4.22 (simplicial arrangement)**.** A central hyperplane arrangement is *simplicial* if all its regions are simplicial cones.

An affine arrangement is simplicial if its associated central arrangement is simplicial.

… corresponds to simple zonotopes. Example: Cube. More interesting example:

Example 4.23. Permutahedron $\Pi_{d-1} = Z(\{e_i - e_j : i < j\}) \subset \mathbb{R}^d$
simple zonotope, $\binom{d}{2}$ zones, given by vectors $e_i - e_j$,
$d!$ vertices, $2^d - 2$ facets.

57

1234

2134   1324   1243

2143

2314   3124   1423

3214   1342   1432

2413   3142

2341   4123

3241   2431   4132

3412   4213

3421   4231   4312

4321

Definition 4.24 (Reflection groups). A *finite reflection group* is a finite group $G \subset O(d)$ that is generated by reflections (i.e., orthogonal reflections in hyperplanes).

The associated *reflection arrangement* is the arrangement of all hyperplanes such that the orthogonal reflection in H is in the group.

Lemma 4.25. *The arrangements of finite reflection groups are simplicial.*

Proof. By Shannon's theorem 4.21, some region is simplicial. On the other hand, all neighbors of a simplicial region are simplicial, as we see from reflections in its walls. $\square$

Definition 4.26. A convex polytope $P \subset \mathbb{R}^d$ is *regular* if the group of symmetries is transitive on the maximal flags, that is, if for any two maximal chain of faces

$$F_0 \subset F_1 \subset \cdots \subset F_{d-1} \subset P \quad \text{and} \quad F_0' \subset F_1' \subset \cdots \subset F_{d-1}' \subset P \quad \text{and}$$

there is an orthogonal transformation that maps F_i to F_i' (and thus in particular P to P, so it is a symmetry of P).

Proposition 4.27. *The symmetry group of a regular polytope is a reflection group.*

The finite reflection groups (in all dimensions) have been classified in the early 20th century, in the connection of Lie theory. Thus it subsumes the classification of (symmetry groups of) regular polytopes, achieved by Ludwig Schläfli around 1845.

Theorem 4.28 (Classification of finite reflection groups (and of regular polytopes)!)**.**

Problem. Challenge: Classify simplicial arrangements, at least in dimension 2.

Is it true that there are only the "obvious ones" (some families derived from the regular n-gons, plus the finite reflection arrangments) and then a finite list of "sporadic" ones?

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