Discrete Geometry I

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— Preliminary Lecture Notes (without any guarantees) —

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This is the first in a series of three courses on Discrete Geometry. We will get to know fascinating geometric structures such as configurations of points and lines, hyperplane arrangements, and in particular polytopes and polyhedra, and learn how to handle them using modern methods for computation and visualization and current analysis and proof techniques. A lot of this looks quite simple and concrete at first sight (and some of it is), but it also very quickly touches topics of current research.

For students with an interest in discrete mathematics and geometry, this is the starting point to specialize in discrete geometry. The topics addressed in the course supplement and deepen the understanding of discrete-geometric structures appearing in differential geometry, optimization, combinatorics, topology, and algebraic geometry. To follow the course, a solid background in linear algebra is necessary. Some knowledge of combinatorics and geometry is helpful.

Basic Literature

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Contents

A rough schedule, which we will adapt as we move along:

1. 0. Introduction/1. Some highlights to start with . . . . . 15. October
2. 2. Basic Structures / 2.1 Convex sets, intersections and separation . . . . . 16. October
3. 2.1.4 Separation theorems; 2.2 Polytopes . . . . . 22. October
4. 2.2.1 Faces and the face lattice . . . . . 23. October
5. 2.2.2 $\mathcal{V}$- and $\mathcal{H}$-polytopes; Representation theorem . . . . . 29. October
6. 2.2.3 Fourier–Motzkin elimination, and the Farkas lemmas . . . . . 30. October
7. 2.3 Polyhedral complexes 2.4 Subdivisions and triangulations . . . . . [?] 5. November
8. 2.5 Configurations of points, hyperplanes . . . . . [?] 6. November
9. 3. Polytope theory . . . . . 12. November
10. . . . . . 13. November
11. . . . . . 19. November
12. . . . . . 20. November
13. . . . . . 26. November
14. . . . . . 27. November
15. . . . . . [?] 3. December
16. . . . . . 4. December
17. . . . . . 10. December
18. . . . . . 11. December
19. . . . . . 17. December
20. . . . . . 18. December
21. 4. Combinatorial Geometry . . . . . [?] 7. January
22. . . . . . [?] 8. January
23. . . . . . 14. January
24. . . . . . 15. January
25. . . . . . 21. January
26. 5. Geometry of linear programming . . . . . 22. January
27. . . . . . 28. January
28. . . . . . 29. January
29. ?? . . . . . [?] 4. February
30. Exam . . . . . [?] 5. February
31. 6. Discrete Geometry Perspectives, I . . . . . 11. February
32. Discrete Geometry Perspectives, II . . . . . 12. February

0 Introduction

What's the goal?

This is a first course in a large and interesting mathematical domain commonly known as "Discrete Geometry". This spans from very classical topics (such as regular polyhedra – see Euclid's *Elements*) to very current research topics (Discrete Geometry, Extremal Geometry, Computational Geometry, Convex Geometry) that are also of great industrial importance (for Computer Graphics, Visualization, Molecular Modelling, and many other topics).

My goal will be to develop these topics in a three-semester sequence of Graduate Courses in such a way that

- you get an **overview** of the field of Discrete Geometry and its manifold connections,
- you learn to understand, analyze, visualize, and confidently/competently argue about the basic **structures** of Discrete Geometry, which includes
  - point configurations/hyperplane arrangements,
  - frameworks
  - subspace arrangements, and
  - polytopes and polyhedra,
- you learn to know (and appreciate) the most important **results** in Discrete Geometry, which includes both simple & basic as well as striking key results,
- you get to learn and practice important **ideas and techniques** from Discrete Geometry (many of which are interesting also for other domains of Mathematics), and
- You learn about current **research topics** and problems treated in Discrete Geometry.

1 Some highlights to start with

1.1 Point configurations

Proposition 1.1 (Sylvester–Gallai 1893/1944)**.** *Every finite set of n points in the plane, not all on a line, n large, defines an "ordinary" line, which contain exactly 2 of the points.*

The "BOOK proof" for this result is due to L. M. Kelly [1].

Theorem/Problem 1.2 (Green–Tao 2012 [4])**.** *Every finite set of n points in the plane, not all on a line, n large, defines at least $n/2$ "ordinary" lines, which contain exactly 2 of the points.*
How large does n have to be for this to be true? $n > 13$?

Theorem/Problem 1.3 (Blagojevic–Matschke–Ziegler 2009 [2])**.** *For $d \geq 1$ and a prime r, any $(r-1)(d+1)+1$ colored points in $\mathbb{R}^d$, where no r points have the same color, can be partitioned into r "rainbow" subsets, in which no 2 points have the same color, such that the convex hulls of the r blocks have a point in common.*
Is this also true if r is not a prime? How about $d = 2$ and $r = 4$, cf. [6]?

1.2 Polytopes

Theorem 1.4 (Schläfli 1852)**.** *The complete classification of regular polytopes in $\mathbb{R}^d$:*
- *d-simplex ($d \geq 1$)*
- *the regular n-gon ($d = 2$, $n \geq 3$)*
- *d-cube and d-crosspolytope ($d \geq 2$)*
- *icosahedron and dodecahedron ($d = 3$)*
- *24-cell ($d = 4$)*
- *120-cell and 600-cell ($d = 4$)*

Theorem/Problem 1.5 (Santos 2012 [9])**.** *There is a simple polytope of dimension $d = 43$ and $n = 86$ facets, whose graph diameter is not, as conjectured by Hirsch (1957), at most 43.*
What is the largest possible graph diameter for a d-dimensional polytope with n facets? Is it a polynomial function of n?

1.3 Sphere configurations/packings/tilings

Theorem/Problem 1.6 (*see* [8])**.** *For $d \geq 2$, the* kissing number *κ_d denotes the maximal number of non-overlapping unit spheres that can simultaneously touch ("kiss") a given unit sphere in $\mathbb{R}^d$.*
$d = 2$: $\kappa_2 = 6$, "hexagon configuration", unique
$d = 3$: $\kappa_3 = 12$, "dodecahedron configuration", not unique
$d = 4$: $\kappa_4 = 24$ (Musin 2008 [7]) "24-cell", unique?
$d = 8$: $\kappa_8 = 240$, E_8 lattice, unique?
$d = 24$: $\kappa_{24} = 196560$, "Leech lattice", unique?

Theorem/Problem 1.7 (Engel 1980 [3] [5] [10]). *There is a stereohedron (that is, a* 3*-dimensional polytope whose congruent copies tile* $\mathbb{R}^3$*) with* 38 *facets. But is the maximal number of facets of a stereohedron in* $\mathbb{R}^3$ *bounded at all?*

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End of class on October 15

2 Basic structures in discrete geometry

2.1 Convex sets, intersections and separation

2.1.1 Convex sets

Geometry in $\mathbb{R}^d$ (or in any finite-dimensional vector space over a real closed field . . .)

Definition 2.1 (Convex set)**.** A set $S \subseteq \mathbb{R}^d$ is *convex* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}_{\geq 0}$, $\lambda + \mu = 1$.

Lemma 2.2. *$S \subseteq \mathbb{R}^d$ is convex if and only if $\sum_{i=1}^k \lambda_i x_i \in S$ for all $k \geq 1$, $x_1, \ldots, x_k \in S$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, $\lambda_1, \ldots, \lambda_k \geq 0$, $\sum_{i=1}^k \lambda_i = 1$.*

Proof. For "if" take the special case $k = 2$.
For "only if" we use induction on k, where the case $k = 1$ is vacuous and $k = 2$ is clear. Without loss of generality, $0 < x_k < 1$. Now rewrite $\sum_{i=1}^k \lambda_i x_i$ as

$$(1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i \; + \; \lambda_k x_k$$

□

Compare:
- $U \subseteq \mathbb{R}^d$ is a *linear subspace* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}$.
- $U \subseteq \mathbb{R}^d$ is an *affine subspace* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}$, $\lambda + \mu = 1$.

2.1.2 Operations on convex sets

Lemma 2.3 (Operations on convex sets)**.** *Let $K, K' \subseteq \mathbb{R}^d$ be convex sets.*
- *$K \cap K' \subseteq \mathbb{R}^d$ is convex.*
- *$K \times K' \subseteq \mathbb{R}^{d+d}$ is convex.*
- *For any affine map $f : \mathbb{R}^d \to \mathbb{R}^e$, $x \mapsto Ax + b$, the image $f(K)$ is convex.*
- *The* Minkowski sum *$K + K' := \{x + y : x \in K,\ y \in K'\}$ is convex.*

Exercise 2.4. Interpret the Minkowski sum as the image of an affine map applied to a product.

Lemma 2.5. *Hyperplanes $H = \{x \in \mathbb{R}^d : a^t x = \alpha\}$ are convex.*
Open halfspaces $H^+ = \{x \in \mathbb{R}^d : a^t x > \alpha\}$ and $H^- = \{x \in \mathbb{R}^d : a^t x < \alpha\}$ are convex.
Closed halfspaces $\overline{H}^+ = \{x \in \mathbb{R}^d : a^t x \geq \alpha\}$ and $\overline{H}^- = \{x \in \mathbb{R}^d : a^t x \leq \alpha\}$ are convex.

More generally, for $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$,
- $\{x \in \mathbb{R}^d : Ax = 0\}$ is a linear subspace,
- $\{x \in \mathbb{R}^d : Ax = b\}$ is an affine subspace,
- $\{x \in \mathbb{R}^d : Ax < b\}$ and $\{x \in \mathbb{R}^d : Ax \leq b\}$ are convex subsets of $\mathbb{R}^d$.

2.1.3 Convex hulls, Radon's lemma and Helly's theorem

Definition 2.6 (convex hull)**.** For any $S \subseteq \mathbb{R}^d$, the *convex hull* of S is defined as

$$\operatorname{conv}(S) := \bigcap \big\{K \subseteq \mathbb{R}^d : K \text{ convex}, \ S \subseteq K \subseteq \mathbb{R}^d\big\}.$$

Note the analogy to the usual definition of *affine hull* (an affine subspace) and *linear hull* (or *span*), a vector subspace.

Exercise 2.7. Show that

- $\operatorname{conv}(S)$ is convex,
- $S \subseteq \operatorname{conv}(S)$,
- $S \subseteq S'$ implies $\operatorname{conv}(S) \subseteq \operatorname{conv}(S')$,
- $\operatorname{conv}(S) = S$ if S is convex, and
- $\operatorname{conv}(\operatorname{conv}(S)) = \operatorname{conv}(S)$.

Lemma 2.8 (Radon's[1] lemma)**.** *Any $d+2$ points $p_1, \dots, p_{d+2} \in \mathbb{R}^d$ can be partitioned into two groups $(p_i)_i \in I$ and $(p_i)_i \notin I$ whose convex hulls intersect.*

Proof. The $d+2$ vectors $\binom{p_1}{1}, \dots, \binom{p_{d+2}}{1} \in \mathbb{R}^{d+1}$ are linearly dependent,

$$\lambda_1 \begin{pmatrix} p_1 \\ 1 \end{pmatrix} + \cdots + \lambda_{d+2} \begin{pmatrix} p_{d+2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here not all λ_i's are zero, so some are positive, some are negative, and we can take $I := \{i : \lambda_i > 0\} \neq \emptyset$. Thus with $\Lambda := \sum_{i \in I} \lambda_i > 0$ we can rewrite the above equation as

$$\sum_{i \in I} \frac{\lambda_i}{\Lambda} p_i = \sum_{i \notin I} \frac{-\lambda_i}{\Lambda} p_i.$$

□

Note that even more so Radon's lemma holds for any $n \geq d+2$ points in $\mathbb{R}^d$.

Theorem 2.9 (Helly's Theorem)**.** *Let $C_1, \dots, C_N$ be a finite family of $N \geq d+1$ convex sets such that any $d+1$ of them have a non-empty intersection. Then the intersection of all N of them is non-empty as well.*

Proof. This is trivial for $N = d+1$. Assume $N \geq d+2$. We use induction on N.
By induction, for each i there is a point $\bar{p}_i$ that lies in all C_j except for possibly C_i. Now form a Radon partition of the points $\bar{p}_i$, and let p be a corresponding intersection point. About this point we find that on the one hand it lies in all C_i except for possibly those with $i \in I$, and on the other hand it lies in all C_i except for possibly those with $i \notin I$. □

Note that the claim of Helly's theorem does not follow if we only require that any d sets intersect (take the C_i to be hyperplanes in general position!) or if we admit infinitely many convex sets (take $C_i := [i, \infty)$).

End of class on October 16

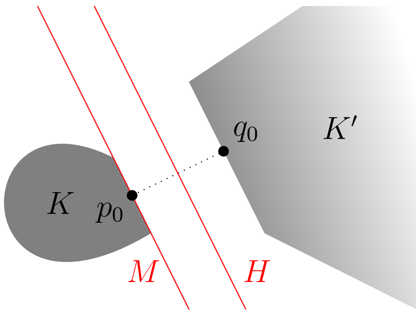
[1] In class, I called this Carathéodory's lemma, which was wrong – Carathéodory's lemma is a related result, which you will see on the problem set.

2.1.4 Separation theorems and supporting hyperplanes

Definition 2.10. A hyperplane H is a *supporting hyperplane* for a convex set K if $K \subset \bar{H}^+$ and $\bar{K} \cap H \neq \emptyset$.

Theorem 2.11 (Separation Theorem)**.** *If $K, K' \neq \emptyset$ are disjoint closed convex sets, where K is compact, then there is a "separating hyperplane" H with $K \subset H^+$ and $K' \subset H^-$.*
Also, in the same situation there is a supporting hyperplane M with $K \subset \overline{M}^+$, $K \cap M \neq \emptyset$, and $K' \subset M^-$.

Proof. Define $\delta := \min\{\|p - q\| : p \in K,\ q \in K'\}$.
The minimum exists, and $\delta > 0$, due to compactness, if we replace K' by an intersection $K' \cap M{\cdot}B^d$ with a large ball, which does not change the result of the minimization.
Furthermore, by compactness there are $p_0 \in K$ and $q_0 \in K'$ with $\|p_0 - q_0\| = \delta$.



Now define H and M' by

$$H := \{x \in \mathbb{R}^d : (p_0 - q_0)^t x = (p_0 - q_0)^t (\tfrac{1}{2} p_0 + \tfrac{1}{2} q_0)\}$$

and

$$M := \{x \in \mathbb{R}^d : (p_0 - q_0)^t x = (p_0 - q_0)^t p_0\}$$

and compute. $\square$

Example 2.12. Consider the (disjoint, closed) convex sets $K := \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$ and $K' := \{(x, y) \in \mathbb{R}^2 : y \geq e^x\}$.

Separation theorems like this are extremely useful not only in Discrete Geometry (as we will see shortly), but also in Optimization. Siehe auch den Hahn–Banach Satz in der Funktionalanalysis.

2.2 Polytopes

Definition 2.13 (Polytope)**.** A *polytope* is the convex hull of a finite set, that is, a subset of the form $P = \operatorname{conv}(S) \subseteq \mathbb{R}^d$ for some finite set $S \subseteq \mathbb{R}^d$.

Examples 2.14. Polytopes: The empty set, any point, any bounded line segment, any triangle, and any convex polygon (in some $\mathbb{R}^n$) is a polytope.

Definition 2.15 (Simplex)**.** Any convex hull of a set of $k+1$ affinely independent points (in $\mathbb{R}^n$, $k \leq n$), is a *simplex*.

Lemma 2.16. *For $p_1, \dots, p_n \in \mathbb{R}^d$, we have*

$$\operatorname{conv}(\{p_1, \dots, p_n\}) = \{\lambda_1 p_1 + \dots + \lambda_n p_n : \lambda_1, \dots, \lambda_n \in \mathbb{R},\ \lambda_1, \dots, \lambda_n \geq 0,\ \lambda_1 + \dots + \lambda_n = 1\}.$$

Proof. For "$\subseteq$" we note that the RHS contains $p_1, \dots, p_n$, and it is convex.
On the other hand, "$\supseteq$" follows from Lemma 2.2. □

Definition 2.17 (Standard simplex)**.** The $(n-1)$-dimensional *standard simplex* in $\mathbb{R}^n$ is

$$\begin{aligned} \Delta_{n-1} &= \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n,\ \lambda_1, \dots, \lambda_n \geq 0,\ \lambda_1 + \dots + \lambda_n = 1\} \\ &= \operatorname{conv}(e_1, \dots, e_n). \end{aligned}$$

Corollary 2.18. *The polytopes are exactly the affine images of the standard simplices.*

Proof. … under the linear (!) map given by $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 p_1 + \dots + \lambda_n p_n$. □

Definition 2.19 (Dimension)**.** The *dimension* of a polytope (and more generally, of a convex set) is defined as the dimension of its affine hull.

Lemma 2.20. *The dimension of* $\operatorname{conv}(\{p_1, \dots, p_n\})$ *is* $\operatorname{rank}\begin{pmatrix} p_1 & \cdots & p_n \\ 1 & \cdots & 1 \end{pmatrix} - 1$.

End of class on October 22

2.2.1 Faces

We are interested in the boundary structure of convex polytopes, as we can describe it in terms of vertices, edges, etc.

Definition 2.21 (Faces)**.** A *face* of a convex polytope P is any subset of the form $F = \{x \in P : a^t x = \alpha\}$, where the linear inequality $a^t x \leq \alpha$ is valid for P (that is, it holds for all $x \in P$).

Thus the empty set $\emptyset$ and the polytope P itself are faces, the *trivial faces*. All other faces are known as the *non-trivial faces*.

Lemma 2.22. *The non-trivial faces F of P are of the form $F = P \cap H$, where H is a supporting hyperplane of P.*

Lemma 2.23. *Every face of a polytope is a polytope.*

Proof. Let $P \coloneqq \operatorname{conv}(S)$ be a polytope and let F be a face of P defined by the inequality $a^t x \leq \alpha$. Define $S_0 \coloneqq \{p \in S : a^t p = \alpha\}$ and $S_- \coloneqq \{p \in S : a^t p < \alpha\}$. Then $S = S_0 \cup S_-$. Now a simple calculation shows that $F = \operatorname{conv}(S_0)$: The convex combination $\lambda_1 p_1 + \dots + \lambda_n p_n$ satisfies the inequality with equality if and only if $\lambda_i = 0$ for all $p_i \in S_-$. To see this, write for example $S_- = \{p_1, \dots, p_k\}$ and $S_0 = \{p'_1, \dots, p'_\ell\}$, and calculate for $x \in F$:

$$\begin{aligned} \alpha = a^t x &= a^t((\lambda_1 p_1 + \dots + \lambda_k p_k) + (\lambda'_1 p'_1 + \dots \lambda'_\ell p'_\ell)) && (1)\\ &= (\lambda_1 a^t p_1 + \dots + \lambda_k a^t p_k) + (\lambda'_1 a^t p'_1 + \dots \lambda'_\ell a^t p'_\ell)) && (2)\\ &\leq (\lambda_1 \alpha + \dots + \lambda_k \alpha) + (\lambda'_1 \alpha + \dots \lambda'_\ell \alpha) && (3)\\ &= \alpha(\lambda_1 + \dots + \lambda_k + \lambda'_1 + \dots \lambda'_\ell) = \alpha, && (4) \end{aligned}$$

where $\lambda_i a^t p_i \leq \lambda_i \alpha$ for $1 \leq i \leq k$ and $\lambda'_j a^t p'_j = \lambda'_j \alpha$ for $1 \leq j \leq \ell$. For this to hold, we must have $\lambda_i a^t p_i = \lambda_i \alpha$, but this holds only if $\lambda_i = 0$ for all i. Thus we have $x = \lambda'_1 p'_1 + \dots \lambda'_\ell p'_\ell$, so $x \in \operatorname{conv}(S_0)$. □

Definition 2.24. Let P be a polytope of dimension d.
The 0-dimensional faces are called *vertices*.
The 1-dimensional faces are called *edges*.
The $(d-2)$-dimensional faces are called *ridges*.
The $(d-1)$-dimensional faces are called *facets*.
A k-dimensional face will also be called a k-face.
The set of all vertices of P is called the *vertex set* of P, denoted $V(P)$.

Proposition 2.25. *Every polytope is the convex hull of its vertex set, $P = \mathrm{conv}(V(P))$. Moreover, if $P = \mathrm{conv}(S)$, then $V(P) \subseteq S$. In particular, every polytope has finitely many vertices.*

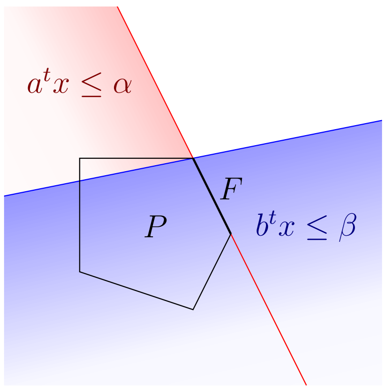
Proof. Let $P = \mathrm{conv}(S)$ and replace S by an inclusion-minimal subset $V = V(P)$ with the property that $P = \mathrm{conv}(V)$. Thus none of the points $p \in V$ are contained in the convex hull of the others, that is, $p \notin \mathrm{conv}(V \backslash p)$. Now the Separation Theorem 2.11, applied to the convex sets $\{p\}$ and $\mathrm{conv}(V \backslash p)$, implies that there is a supporting hyperplane for $\{p\}$ (that is, a hyperplane through p) which does not meet $\mathrm{conv}(V \backslash p)$.
We take the corresponding linear inequality, which is satisfied by p with equality, and by all points in $\mathrm{conv}(V \backslash p)$ strictly. Thus $\{p\}$ is a face: a vertex. $\square$

Proposition 2.26. *Every face of a face of P is a face of P.*

Proof. Let $F \subset P$ be a face, defined by $a^t x \leq \alpha$. Let $G \subset F$ be a face, defined by $b^t x \leq \beta$. Then for sufficiently small $\varepsilon > 0$, the inequality

$$(a + \varepsilon b)^t x \leq \alpha + \varepsilon \beta$$

is strictly satisfied for all vertices in $V(P) \backslash F$, since this is strictly satisfied for $\varepsilon = 0$, so this leads to finitely-many conditions for ε to be "small enough." It is also strictly satisfied on $F \setminus G$ if $\varepsilon > 0$, and it is satisfied with equality on G.



Now let x be any point in $P \setminus F$. Then we can write x as a convex combination of the vertices in P, say

$$x = (\lambda_1 v_1 + \cdots + \lambda_k v_k) + (\lambda'_1 v'_1 + \ldots \lambda'_\ell v'_\ell)$$

for $S_- = \{v_1, \ldots, v_k\}$ and $S_0 = \{v'_1, \ldots, v'_\ell\}$ as in the proof of Lemma 2.23. As x does not lie in F, the coefficient of at least one vertex v_i of P not in F is positive. This implies that the inequality displayed above is strict for x. $\square$

Corollary 2.27. *Every face F of a polytope P is the convex hull of the vertices of P that are contained in F:*

$$V(F) = F \cap V(P).$$

Proof. "$\subseteq$" is from Proposition 2.26. "$\supseteq$" is trivial. $\square$

End of class on October 23

In particular, any polytope has only finitely many faces.

Lemma 2.28. *Any intersection of faces of a polytope P is a face of P.*

Proof. Add the inequalities. $\square$

Definition 2.29 (Vertex figure)**.** Let v be a vertex of a d-dimensional polytope P, and let H be a hyperplane that separates v from $\mathrm{conv}(V(P) \setminus \{v\})$. Then

$$P/v := P \cap H$$

is called a *vertex figure* of P at v.

Proposition 2.30. *If $P = \mathrm{conv}(S \cup \{v\})$ with $a^t v > \alpha$ while $a^s < \alpha$ for $s \in S$, where $H = \{x \in \mathbb{R}^d : a^t = \alpha\}$, then*

$$P/v = \mathrm{conv}\Big\{ \frac{a^t v - \alpha}{a^t v - a^t s} s + \frac{\alpha - a^t s}{a^t v - a^t s} v : s \in S \Big\}.$$

In particular, P/v is a polytope.

Proof. "$\supseteq$": the points $\bar{s} := \frac{a^t v - \alpha}{a^t v - a^t s} s + \frac{\alpha - a^t s}{a^t v - a^t s} v$ have been constructed as points $\lambda s + (1-\lambda) v$ such that $a^t \bar{s} = \alpha$, so $\bar{s} \in P/v$.
"$\subseteq$": *calculate* that if $x \in \mathrm{conv}(S \cup \{v\})$ satisfies $a^t x = \alpha$, then it can be written as a convex combination of the points $\bar{s}$. For this, write

$$\begin{aligned}
x &= \sum_i \lambda_i s_i + \lambda_0 v \\
&= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \frac{a^t v - \alpha}{a^t v - a^t s_i} s_i + \lambda_0 v \\
&= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \Big(\frac{a^t v - \alpha}{a^t v - a^t s_i} s_i + \frac{\alpha - a^t s_i}{a^t v - a^t s_i} v \Big) + \Big(\lambda_0 - \sum_i \lambda_i \frac{\alpha - a^t s_i}{a^t v - \alpha} \Big) v \\
&= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \bar{s}_i + \Big(\lambda_0 - \frac{\alpha \sum_i \lambda_i - \sum_i \lambda_i a^t s_i}{a^t v - \alpha} \Big) v.
\end{aligned}$$

At this point we use that $x \in H$, that is, $a^t x = \sum_i \lambda_i a^t s_i + \lambda_0 a^t v = \alpha$, and that this was a convex combination, so $\sum_i \lambda_i = 1 - \lambda_0$, to conclude that the last term in large parentheses is 0. $\square$

Exercise 2.31. Let $P := \mathrm{conv}\{\pi(\pm 1, \pm 1, 0, 0) : \pi \in \mathfrak{S}_4\}$ be the convex hull of all the vectors that have two ± 1 entries and two zero coordinates.

- How many vectors are these?
- Why are they all vertices?
- Why do they all have the same vertex figure?
- Compute one vertex figure.

Proposition 2.32. *For any vertex v of a d-polytope P, the k-dimensional faces of P/v are in an inclusion-preserving bijection with the $(k+1)$-dimensional faces of P that contain v.*
In particular, P/v is a polytope of dimension $d-1$.

Proof. Clearly if F is a face of P, then $F \cap H$ is a face of $P \cap H = P/v$.
Note that $v \notin H$. Thus every $(k+1)$-face $F \subseteq P$ with $v \in P$ defines a k-face F/v of P/v: From the previous proof we can see that $\operatorname{aff}((F \cap H) \cup \{v\}) = \operatorname{aff}(F)$.
For the converse, let $G \subseteq P/v$ be a k-face, defined by the inequality $b^t x \leq \beta$. Then we calculate that this inequality, plus a suitable (not necessarily positive!) multiple of the equation $a^t x = \alpha$ defining H, is satisfied with equality on $P \cap (\operatorname{aff}(G \cup \{v\}))$, but strictly on all other points of P. Explicitly, the inequality we consider is

$$(b^t + \mu a^t)x \leq \beta + \mu\alpha, \tag{5}$$

and this will be satisfied with equality on v if $(b^t + \mu a^t)v = \beta + \mu\alpha$, that is, if $\mu = -\frac{b^t v - \beta}{a^t v - \alpha}$, where the denominator is positive. This inequality (5) is valid on P/v and valid with equality on v. Let $P = \operatorname{conv}(S \cup \{v\}$. Then the inequality is valid on all points of S as well, since a point $s \in S$ that violates it would give rise to $\bar{s} \in P/v$ that violates it as well.
Thus

$$\widehat{G} := P \cap (\operatorname{aff}(G \cup \{v\}))$$

is the desired $(k+1)$-face of P. $\square$

End of class on October 29

2.2.2 Order theory and the face lattice

Definition 2.33 (Posets and lattices)**.** A *poset* is a partially ordered set, that is, a set S with a binary relation "$\leq$" that is *reflexive* ($x \leq x$ for all $x \in S$), *asymmetric* ($x \leq y \leq x$ implies $x = y$) and *transitive* ($x \leq y \leq z$ implies $x \leq z$). (All posets we consider are finite.) Formally, the poset could be written $(S, \leq)$, but it is customary to write the same letter S for the poset.
An *interval* in a poset $(S, \leq)$ is a subposet (i.e., a subset with the induced partial order) of the form

$$[x, y] := (\{z \in S : x \leq z \leq y\}, \leq)$$

for $x, y \in S$, $x \leq y$.
A *chain* in a poset is a totally-ordered subset.
A poset is *bounded* if it has a unique minimal element, denoted $\hat{0}$, and a unique maximal element, denoted $\hat{1}$.
A poset is *graded* if it has a unique minimal element $\hat{0}$, and if for every element x of the poset, all maximal chains from $\hat{0}$ to x have the same length, called the *rank* of the element, usually

denoted $r(x)$. The function $r : S \to \mathbb{N}_0$ is then called the *rank function* of S. If a poset is graded and has a maximal element $\hat{1}$, we write $r(S) := r(\hat{1})$ for the *rank of the poset*.

A poset is a *lattice* if any two elements a, b have a unique minimal upper bound, denoted $a \vee b$, called the *join* of a and b, and a unique maximal lower bound, denoted $a \wedge b$, and called the *meet* of a and b.

Exercise 2.34. Let $(Q, \leq)$ be a finite partial order. Show that any two of the following properties yield the third:

1. The poset is bounded.
2. Meets exist.
3. Joins exist.

Exercise 2.35. Let Q be a finite lattice, and A be an arbitrary subset. Then A has a unique minimal upper bound, the *join* $\bigvee A$, and a unique maximal lower bound, the *meet* $\bigwedge A$.

Theorem 2.36 (The polytope face lattice)**.** *The face poset* $(\mathcal{F}, \subset)$ *of any polytope is a finite graded lattice, denoted* $L = L(P)$*, of rank* $r(L(P)) = \dim(P) + 1$.

Proof. This is a finite bounded poset, with minimal element $\hat{0} = \emptyset$ and maximal element $\hat{1} = P$. Meet exists, as clearly $F \wedge F' = F \cap F'$ is the largest face contained in both F and F'. (The intersection is a face by Lemma 2.28.) Thus $L(P)$ is a lattice.

If $G \subset F$ are faces, then in particular G is a face of F, and thus $\dim(G) < \dim(F)$. Thus all we have to prove is that if $\dim(F) \geq \dim(G) + 2$, then there is a face H with $G \subset H \subset F$.

If $F \subset P$, then $\dim(F) < \dim(P)$, so we are done by induction.

If $\emptyset \subset G$, then G has a vertex v, and $[G, F] \subseteq [v, P] = L(P/v)$, where $\dim(P/v) < \dim(P)$, so we are done by induction.

If $G = \emptyset$ and $F = P$, where $\dim(P) \geq 1$, then P has a vertex w, where $\emptyset \subset \{w\} \subset P$. $\square$

Definition 2.37 (Combinatorially equivalent)**.** Two polytopes P and P' are *combinatorially equivalent* if their face lattices L, L' are isomorphic as posets, that is, if there is a bijection $f : L \to L'$ such that $x \leq_L y$ holds in P if and only if $f(x) \leq_{L'} f(y)$ holds in P'.

Exercise 2.38. Define "isomorphic" for posets, and for lattices. Show that if Q is a poset and L is a lattice, and if Q and L are isomorphic as posets, then Q is a lattice, and Q and L are also isomorphic as lattices.

Exercise 2.39. Let us consider the poset $D(n)$ of all divisors of the natural number n (examples to try: 24 and 42 and 64), ordered by divisibility. Are these posets? Are they bounded? Are they lattices? Graded? What is the rank function? Can you describe join and meet?

For which n is there a polytope with $D(n)$ isomorphic to its face lattice?

Lemma 2.40. *If two polytopes* P, P' *are affinely isomorphic (that is, if there is an affine bijective map* $P \to P'$*), then they are combinatorially equivalent. The converse is wrong.*

Lemma 2.41 (Face lattice of a simplex)**.** *Let* Δ_{k-1} *be a* $(k-1)$*-dimensional simplex (with* k *vertices). Its face lattice is isomorphic to the poset of all subsets of a* k*-element set, ordered by inclusion known as the* Boolean algebra B_k *of rank* k*, as given for example by* $(2^{[k]}, \subseteq)$*, where* $2^{[k]}$ *denotes the collection of all subsets of* $[k] := \{1, 2, \ldots, k\}$.

Proof. Any two $(k-1)$-simplices are affinely equivalent.
Any subset of the vertex set of a simplex defines a face, which is a simplex. $\square$

Exercise 2.42. Prove that if any subset of vertex set of a polytope defines a face, then the polytope is a simplex.

Theorem 2.43 (Intervals in polytope face lattices)**.** *Let $G \subseteq F$ be faces of a polytope P. Then the interval*

$$[G, F] = (\{H \in L(P) : G \subseteq H \subseteq F\}, \subseteq)$$

of $L(P)$ is the face lattice of a polytope of dimension $\dim(F) - \dim(G) - 1$.
In particular, if $G = \emptyset$, then $[G, F] = L(F)$.
In particular, if $F = P$ and $G = \{v\}$ is a vertex, then $[G, F] = L(P/v)$.

Proof. The two "in particular" statements follow from Propositions 2.26 and 2.32. Now we can use induction. $\square$

Corollary 2.44 (Diamond property)**.** *Any interval $[x, y]$ of length 2 in a polytope face lattice contains exactly two elements z with $x < z < y$.*

This "harmless lemma" has substantial consequences.

Corollary 2.45. *For every polytope, every face is the minimal face containing a certain set of vertices. (More precisely, every face is the convex hull of the vertices it contains.)*
Simultaneously, every face is an intersection of facets (it is the intersection of the facets it is contained in).

Proof. This says that every element in the face lattice of a polytope is a join of vertices, and a meet of facets.
This can be phrased and proved entirely in lattice-theoretic language: Take a graded lattice of rank $d+1$ with the diamond property. Then every element of rank $r(x) \leq d$ is a meet of elements of rank $d-1$ (which would be called "co-atoms"). Simultaneously, every element of rank $r(x) > 0$ is a join of elements of rank 1 (which are called "atoms").
To prove this, note that for an element of rank $k \geq 2$ the diamond property shows that it is the join of two elements of rank $k-1$, and by induction those are joins of atoms. Dually for meets of coatoms. $\square$

End of class on October 30