Discrete Geometry I

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— Preliminary Lecture Notes (without any guarantees) —

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This is the first in a series of three courses on Discrete Geometry. We will get to know fascinating geometric structures such as configurations of points and lines, hyperplane arrangements, and in particular polytopes and polyhedra, and learn how to handle them using modern methods for computation and visualization and current analysis and proof techniques. A lot of this looks quite simple and concrete at first sight (and some of it is), but it also very quickly touches topics of current research.

For students with an interest in discrete mathematics and geometry, this is the starting point to specialize in discrete geometry. The topics addressed in the course supplement and deepen the understanding of discrete-geometric structures appearing in differential geometry, optimization, combinatorics, topology, and algebraic geometry. To follow the course, a solid background in linear algebra is necessary. Some knowledge of combinatorics and geometry is helpful.

Basic Literature

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0 Introduction

What's the goal?

This is a first course in a large and interesting mathematical domain commonly known as "Discrete Geometry". This spans from very classical topics (such as regular polyhedra – see Euclid's *Elements*) to very current research topics (Discrete Geometry, Extremal Geometry, Computational Geometry, Convex Geometry) that are also of great industrial importance (for Computer Graphics, Visualization, Molecular Modelling, and many other topics).

My goal will be to develop these topics in a three-semester sequence of Graduate Courses in such a way that

- you get an **overview** of the field of Discrete Geometry and its manifold connections,
- you learn to understand, analyze, visualize, and confidently/competently argue about the basic **structures** of Discrete Geometry, which includes
  - point configurations/hyperplane arrangements,
  - frameworks
  - subspace arrangements, and
  - polytopes and polyhedra,
- you learn to know (and appreciate) the most important **results** in Discrete Geometry, which includes both simple & basic as well as striking key results,
- you get to learn and practice important **ideas and techniques** from Discrete Geometry (many of which are interesting also for other domains of Mathematics), and
- You learn about current **research topics** and problems treated in Discrete Geometry.

1 Some highlights to start with

1.1 Point configurations

Proposition 1.1 (Sylvester–Gallai 1893/1944)**.** *Every finite set of n points in the plane, not all on a line, n large, defines an "ordinary" line, which contain exactly 2 of the points.*

The "BOOK proof" for this result is due to L. M. Kelly [1].

Theorem/Problem 1.2 (Green–Tao 2012 [4])**.** *Every finite set of n points in the plane, not all on a line, n large, defines at least $n/2$ "ordinary" lines, which contain exactly 2 of the points.*
How large does n have to be for this to be true? $n > 13$?

Theorem/Problem 1.3 (Blagojevic–Matschke–Ziegler 2009 [2])**.** *For $d \geq 1$ and a prime r, any $(r-1)(d+1)+1$ colored points in $\mathbb{R}^d$, where no r points have the same color, can be partitioned into r "rainbow" subsets, in which no 2 points have the same color, such that the convex hulls of the r blocks have a point in common.*
Is this also true if r is not a prime? How about $d = 2$ and $r = 4$, cf. [6]?

1.2 Polytopes

Theorem 1.4 (Schläfli 1852)**.** *The complete classification of regular polytopes in $\mathbb{R}^d$:*
– *d-simplex ($d \geq 1$)*
– *the regular n-gon ($d = 2$, $n \geq 3$)*
– *d-cube and d-crosspolytope ($d \geq 2$)*
– *icosahedron and dodecahedron ($d = 3$)*
– *24-cell ($d = 4$)*
– *120-cell and 600-cell ($d = 4$)*

Theorem/Problem 1.5 (Santos 2012 [9])**.** *There is a simple polytope of dimension $d = 43$ and $n = 86$ facets, whose graph diameter is not, as conjectured by Hirsch (1957), at most 43.*
What is the largest possible graph diameter for a d-dimensional polytope with n facets? Is it a polynomial function of n?

1.3 Sphere configurations/packings/tilings

Theorem/Problem 1.6 (*see* [8])**.** *For $d \geq 2$, the* kissing number *κ_d denotes the maximal number of non-overlapping unit spheres that can simultaneously touch ("kiss") a given unit sphere in $\mathbb{R}^d$.*
$d = 2$: $\kappa_2 = 6$, "hexagon configuration", unique
$d = 3$: $\kappa_3 = 12$, "dodecahedron configuration", not unique
$d = 4$: $\kappa_4 = 24$ (Musin 2008 [7]) *"24-cell", unique?*
$d = 8$: $\kappa_8 = 240$, E_8 lattice, unique?
$d = 24$: $\kappa_{24} = 196560$, "Leech lattice", unique?

Theorem/Problem 1.7 (Engel 1980 [3] [5] [10])**.** *There is a stereohedron (that is, a* 3*-dimensional polytope whose congruent copies tile* $\mathbb{R}^3$*) with* 38 *facets. But is the maximal number of facets of a stereohedron in* $\mathbb{R}^3$ *bounded at all?*

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End of class on October 15

2 Basic structures in discrete geometry

2.1 Convex sets, intersections and separation

2.1.1 Convex sets

Geometry in $\mathbb{R}^d$ (or in any finite-dimensional vector space over a real closed field ...)

Definition 2.1 (Convex set)**.** A set $S \subseteq \mathbb{R}^d$ is *convex* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}_{\geq 0}$, $\lambda + \mu = 1$.

Lemma 2.2. *$S \subseteq \mathbb{R}^d$ is convex if and only if $\sum_{i=1}^{k} \lambda_i x_i \in S$ for all $k \geq 1$, $x_1, \dots, x_k \in S$, $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, $\lambda_1, \dots, \lambda_k \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$.*

Proof. For "if" take the special case $k = 2$.
For "only if" we use induction on k, where the case $k = 1$ is vacuous and $k = 2$ is clear. Without loss of generality, $0 < x_k < 1$. Now rewrite $\sum_{i=1}^{k} \lambda_i x_i$ as

$$(1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i \; + \; \lambda_k x_k$$

□

Compare:

- $U \subseteq \mathbb{R}^d$ is a *linear subspace* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}$.
- $U \subseteq \mathbb{R}^d$ is an *affine subspace* if $\lambda p + \mu q \in S$ for all $p, q \in S$, $\lambda, \mu \in \mathbb{R}$, $\lambda + \mu = 1$.

2.1.2 Operations on convex sets

Lemma 2.3 (Operations on convex sets)**.** *Let $K, K' \subseteq \mathbb{R}^d$ be convex sets.*

- *$K \cap K' \subseteq \mathbb{R}^d$ is convex.*
- *$K \times K' \subseteq \mathbb{R}^{d+d}$ is convex.*
- *For any affine map $f : \mathbb{R}^d \to \mathbb{R}^e$, $x \mapsto Ax + b$, the image $f(K)$ is convex.*
- *The* Minkowski sum *$K + K' := \{x + y : x \in K, \ y \in K'\}$ is convex.*

Exercise 2.4. Interpret the Minkowski sum as the image of an affine map applied to a product.

Lemma 2.5. *Hyperplanes $H = \{x \in \mathbb{R}^d : a^t x = \alpha\}$ are convex.*
Open halfspaces $H^+ = \{x \in \mathbb{R}^d : a^t x > \alpha\}$ and $H^- = \{x \in \mathbb{R}^d : a^t x < \alpha\}$ are convex.
Closed halfspaces $\overline{H}^+ = \{x \in \mathbb{R}^d : a^t x \geq \alpha\}$ and $\overline{H}^- = \{x \in \mathbb{R}^d : a^t x \leq \alpha\}$ are convex.

More generally, for $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$,

- $\{x \in \mathbb{R}^d : Ax = 0\}$ is a linear subspace,
- $\{x \in \mathbb{R}^d : Ax = b\}$ is an affine subspace,
- $\{x \in \mathbb{R}^d : Ax < b\}$ and $\{x \in \mathbb{R}^d : Ax \leq b\}$ are convex subsets of $\mathbb{R}^d$.

2.1.3 Convex hulls, Radon's lemma and Helly's theorem

Definition 2.6 (convex hull)**.** For any $S \subseteq \mathbb{R}^d$, the *convex hull* of S is defined as

$$\operatorname{conv}(S) := \bigcap \{K \subseteq \mathbb{R}^d : K \text{ convex}, \ S \subseteq K \subseteq \mathbb{R}^d\}.$$

Note the analogy to the usual definition of *affine hull* (an affine subspace) and *linear hull* (or *span*), a vector subspace.

Exercise 2.7. Show that

- $\operatorname{conv}(S)$ is convex,
- $S \subseteq \operatorname{conv}(S)$,
- $S \subseteq S'$ implies $\operatorname{conv}(S) \subseteq \operatorname{conv}(S')$,
- $\operatorname{conv}(S) = S$ if S is convex, and
- $\operatorname{conv}(\operatorname{conv}(S)) = \operatorname{conv}(S)$.

Lemma 2.8 (Radon's[1] lemma)**.** *Any $d+2$ points $p_1, \ldots, p_{d+2} \in \mathbb{R}^d$ can be partitioned into two groups $(p_i)_i \in I$ and $(p_i)_i \notin I$ whose convex hulls intersect.*

Proof. The $d+2$ vectors $\binom{p_1}{1}, \ldots, \binom{p_{d+2}}{1} \in \mathbb{R}^{d+1}$ are linearly dependent,

$$\lambda_1 \begin{pmatrix} p_1 \\ 1 \end{pmatrix} + \cdots + \lambda_{d+2} \begin{pmatrix} p_{d+2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here not all λ_i's are zero, so some are positive, some are negative, and we can take $I := \{i : \lambda_i > 0\} \neq \emptyset$. Thus with $\Lambda := \sum_{i \in I} \lambda_i > 0$ we can rewrite the above equation as

$$\sum_{i \in I} \frac{\lambda_i}{\Lambda} p_i = \sum_{i \notin I} \frac{-\lambda_i}{\Lambda} p_i.$$

□

Note that even more so Radon's lemma holds for any $n \geq d+2$ points in $\mathbb{R}^d$.

Theorem 2.9 (Helly's Theorem)**.** *Let $C_1, \ldots, C_N$ be a finite family of $N \geq d+1$ convex sets such that any $d+1$ of them have a non-empty intersection. Then the intersection of all N of them is non-empty as well.*

Proof. This is trivial for $N = d+1$. Assume $N \geq d+2$. We use induction on N.
By induction, for each i there is a point $\bar{p}_i$ that lies in all C_j except for possibly C_i. Now form a Radon partition of the points $\bar{p}_i$, and let p be a corresponding intersection point. About this point we find that on the one hand it lies in all C_i except for possibly those with $i \in I$, and on the other hand it lies in all C_i except for possibly those with $i \notin I$. □

Note that the claim of Helly's theorem does not follow if we only require that any d sets intersect (take the C_i to be hyperplanes in general position!) or if we admit infinitely many convex sets (take $C_i := [i, \infty)$).

End of class on October 16

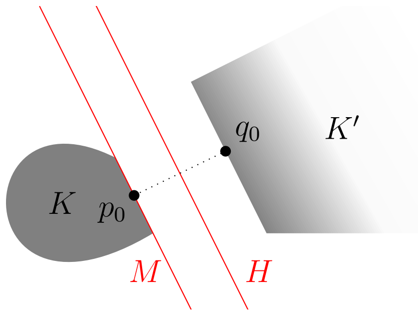
[1] In class, I called this Carathéodory's lemma, which was wrong – Carathéodory's lemma is a related result, which you will see on the problem set.

2.1.4 Separation theorems and supporting hyperplanes

Definition 2.10. A hyperplane H is a *supporting hyperplane* for a convex set K if $K \subset \bar{H}^+$ and $\bar{K} \cap H \neq \emptyset$.

Theorem 2.11 (Separation Theorem)**.** *If $K, K' \neq \emptyset$ are disjoint closed convex sets, where K is compact, then there is a "separating hyperplane" H with $K \subset H^+$ and $K' \subset H^-$.*
Also, in the same situation there is a supporting hyperplane M with $K \subset \overline{M}^+$, $K \cap M \neq \emptyset$, and $K' \subset M^-$.

Proof. Define $\delta := \min\{\|p - q\| : p \in K,\, q \in K'\}$.
The minimum exists, and $\delta > 0$, due to compactness, if we replace K' by an intersection $K' \cap M{\cdot}B^d$ with a large ball, which does not change the result of the minimization.
Furthermore, by compactness there are $p_0 \in K$ and $q_0 \in K'$ with $\|p_0 - q_0\| = \delta$.



Now define H and M' by

$$H := \{x \in \mathbb{R}^d : (p_0 - q_0)^t x = (p_0 - q_0)^t (\tfrac{1}{2} p_0 + \tfrac{1}{2} q_0)\}$$

and

$$M := \{x \in \mathbb{R}^d : (p_0 - q_0)^t x = (p_0 - q_0)^t p_0\}$$

and compute. □

Example 2.12. Consider the (disjoint, closed) convex sets $K := \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$ and $K' := \{(x, y) \in \mathbb{R}^2 : y \geq e^x\}$.

Separation theorems like this are extremely useful not only in Discrete Geometry (as we will see shortly), but also in Optimization. Siehe auch den Hahn–Banach Satz in der Funktionalanalysis.

2.2 Polytopes

Definition 2.13 (Polytope)**.** A *polytope* is the convex hull of a finite set, that is, a subset of the form $P = \operatorname{conv}(S) \subseteq \mathbb{R}^d$ for some finite set $S \subseteq \mathbb{R}^d$.

Examples 2.14. Polytopes: The empty set, any point, any bounded line segment, any triangle, and any convex polygon (in some $\mathbb{R}^n$) is a polytope.

Definition 2.15 (Simplex)**.** Any convex hull of a set of $k+1$ affinely independent points (in $\mathbb{R}^n$, $k \leq n$), is a *simplex.*

Lemma 2.16. *For $p_1, \dots, p_n \in \mathbb{R}^d$, we have*

$$\operatorname{conv}(\{p_1, \dots, p_n\}) = \{\lambda_1 p_1 + \dots + \lambda_n p_n : \lambda_1, \dots, \lambda_n \in \mathbb{R},\ \lambda_1, \dots, \lambda_n \geq 0,\ \lambda_1 + \dots + \lambda_n = 1\}.$$

Proof. For "$\subseteq$" we note that the RHS contains $p_1, \dots, p_n$, and it is convex.
On the other hand, "$\supseteq$" follows from Lemma 2.2. □

Definition 2.17 (Standard simplex)**.** The $(n-1)$-dimensional *standard simplex* in $\mathbb{R}^n$ is

$$\begin{aligned}\Delta_{n-1} &= \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n,\ \lambda_1, \dots, \lambda_n \geq 0,\ \lambda_1 + \dots + \lambda_n = 1\} \\ &= \operatorname{conv}\{e_1, \dots, e_n\}.\end{aligned}$$

Corollary 2.18. *The polytopes are exactly the affine images of the standard simplices.*

Proof. ... under the linear (!) map given by $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 p_1 + \dots + \lambda_n p_n$. □

Definition 2.19 (Dimension)**.** The *dimension* of a polytope (and more generally, of a convex set) is defined as the dimension of its affine hull.

Lemma 2.20. *The dimension of* $\operatorname{conv}(\{p_1, \dots, p_n\})$ *is* $\operatorname{rank}\begin{pmatrix} p_1 & \cdots & p_n \\ 1 & \cdots & 1 \end{pmatrix} - 1$.

End of class on October 22

2.2.1 Faces

We are interested in the boundary structure of convex polytopes, as we can describe it in terms of vertices, edges, etc.

Definition 2.21 (Faces)**.** A *face* of a convex polytope P is any subset of the form $F = \{x \in P : a^t x = \alpha\}$, where the linear inequality $a^t x \leq \alpha$ is valid for P (that is, it holds for all $x \in P$).

Thus the empty set $\emptyset$ and the polytope P itself are faces, the *trivial faces*. All other faces are known as the *non-trivial faces*.

Lemma 2.22. *The non-trivial faces F of P are of the form $F = P \cap H$, where H is a supporting hyperplane of P.*

Lemma 2.23. *Every face of a polytope is a polytope.*

Proof. Let $P := \operatorname{conv}(S)$ be a polytope and let F be a face of P defined by the inequality $a^t x \leq \alpha$. Define $S_0 := \{p \in S : a^t p = \alpha\}$ and $S_- := \{p \in S : a^t p < \alpha\}$. Then $S = S_0 \cup S_-$. Now a simple calculation shows that $F = \operatorname{conv}(S_0)$: The convex combination $\lambda_1 p_1 + \dots + \lambda_n p_n$ satisfies the inequality with equality if and only if $\lambda_i = 0$ for all $p_i \in S_-$. To see this, write for example $S_- = \{p_1, \dots, p_k\}$ and $S_0 = \{p'_1, \dots, p'_\ell\}$, and calculate for $x \in F$:

$$\begin{aligned}\alpha = a^t x &= a^t((\lambda_1 p_1 + \dots + \lambda_k p_k) + (\lambda'_1 p'_1 + \dots \lambda'_\ell p'_\ell)) && (1)\\ &= (\lambda_1 a^t p_1 + \dots + \lambda_k a^t p_k) + (\lambda'_1 a^t p'_1 + \dots \lambda'_\ell a^t p'_\ell)) && (2)\\ &\leq (\lambda_1 \alpha + \dots + \lambda_k \alpha) + (\lambda'_1 \alpha + \dots \lambda'_\ell \alpha) && (3)\\ &= \alpha(\lambda_1 + \dots + \lambda_k + \lambda'_1 + \dots \lambda'_\ell) = \alpha, && (4)\end{aligned}$$

where $\lambda_i a^t p_i \leq \lambda_i \alpha$ for $1 \leq i \leq k$ and $\lambda'_j a^t p'_j = \lambda'_j \alpha$ for $1 \leq j \leq \ell$. For this to hold, we must have $\lambda_i a^t p_i = \lambda_i \alpha$, but this holds only if $\lambda_i = 0$ for all i. Thus we have $x = \lambda'_1 p'_1 + \dots \lambda'_\ell p'_\ell$, so $x \in \operatorname{conv}(S_0)$. □

Definition 2.24. Let P be a polytope of dimension d.
The 0-dimensional faces are called *vertices*.
The 1-dimensional faces are called *edges*.
The $(d-2)$-dimensional faces are called *ridges*.
The $(d-1)$-dimensional faces are called *facets*.
A k-dimensional face will also be called a k-face.
The set of all vertices of P is called the *vertex set* of P, denoted $V(P)$.

Proposition 2.25. *Every polytope is the convex hull of its vertex set, $P = \operatorname{conv}(V(P))$. Moreover, if $P = \operatorname{conv}(S)$, then $V(P) \subseteq S$. In particular, every polytope has finitely many vertices.*

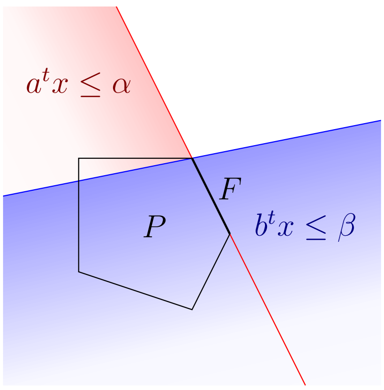
Proof. Let $P = \operatorname{conv}(S)$ and replace S by an inclusion-minimal subset $V = V(P)$ with the property that $P = \operatorname{conv}(V)$. Thus none of the points $p \in V$ are contained in the convex hull of the others, that is, $p \notin \operatorname{conv}(V \backslash p)$. Now the Separation Theorem 2.11, applied to the convex sets $\{p\}$ and $\operatorname{conv}(V \backslash p)$, implies that there is a supporting hyperplane for $\{p\}$ (that is, a hyperplane through p) which does not meet $\operatorname{conv}(V \backslash p)$.
We take the corresponding linear inequality, which is satisfied by p with equality, and by all points in $\operatorname{conv}(V \backslash p)$ strictly. Thus $\{p\}$ is a face: a vertex. □

Proposition 2.26. *Every face of a face of P is a face of P.*

Proof. Let $F \subset P$ be a face, defined by $a^t x \leq \alpha$. Let $G \subset F$ be a face, defined by $b^t x \leq \beta$. Then for sufficiently small $\varepsilon > 0$, the inequality

$$(a + \varepsilon b)^t x \leq \alpha + \varepsilon\beta$$

is strictly satisfied for all vertices in $V(P) \backslash F$, since this is strictly satisfied for $\varepsilon = 0$, so this leads to finitely-many conditions for ε to be "small enough." It is also strictly satisfied on $F \setminus G$ if $\varepsilon > 0$, and it is satisfied with equality on G.



Now let x be any point in $P \setminus F$. Then we can write x as a convex combination of the vertices in P, say

$$x = (\lambda_1 v_1 + \cdots + \lambda_k v_k) + (\lambda'_1 v'_1 + \ldots \lambda'_\ell v'_\ell)$$

for $S_- = \{v_1, \ldots, v_k\}$ and $S_0 = \{v'_1, \ldots, v'_\ell\}$ as in the proof of Lemma 2.23. As x does not lie in F, the coefficient of at least one vertex v_i of P not in F is positive. This implies that the inequality displayed above is strict for x. □

Corollary 2.27. *Every face F of a polytope P is the convex hull of the vertices of P that are contained in F:*

$$V(F) = F \cap V(P).$$

Proof. "$\subseteq$" is from Proposition 2.26. "$\supseteq$" is trivial. $\square$

End of class on October 23

In particular, any polytope has only finitely many faces.

Lemma 2.28. *Any intersection of faces of a polytope P is a face of P.*

Proof. Add the inequalities. $\square$

Definition 2.29 (Vertex figure)**.** Let v be a vertex of a d-dimensional polytope P, and let H be a hyperplane that separates v from $\operatorname{conv}(V(P) \setminus \{v\})$. Then

$$P/v := P \cap H$$

is called a *vertex figure* of P at v.

Proposition 2.30. *If $P = \operatorname{conv}(S \cup \{v\})$ with $a^t v > \alpha$ while $a^s < \alpha$ for $s \in S$, where $H = \{x \in \mathbb{R}^d : a^t = \alpha\}$, then*

$$P/v = \operatorname{conv}\Big\{ \frac{a^t v - \alpha}{a^t v - a^t s} s + \frac{\alpha - a^t s}{a^t v - a^t s} v : s \in S \Big\}.$$

In particular, P/v is a polytope.

Proof. "$\supseteq$": the points $\bar{s} := \frac{a^t v - \alpha}{a^t v - a^t s} s + \frac{\alpha - a^t s}{a^t v - a^t s} v$ have been constructed as points $\lambda s + (1-\lambda)v$ such that $a^t \bar{s} = \alpha$, so $\bar{s} \in P/v$.
"$\subseteq$": *calculate* that if $x \in \operatorname{conv}(S \cup \{v\})$ satisfies $a^t x = \alpha$, then it can be written as a convex combination of the points $\bar{s}$. For this, write

$$\begin{aligned} x &= \sum_i \lambda_i s_i + \lambda_0 v \\ &= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \frac{a^t v - \alpha}{a^t v - a^t s_i} s_i + \lambda_0 v \\ &= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \Big(\frac{a^t v - \alpha}{a^t v - a^t s_i} s_i + \frac{\alpha - a^t s_i}{a^t v - a^t s_i} v \Big) \ + \ \Big(\lambda_0 - \sum_i \lambda_i \frac{\alpha - a^t s_i}{a^t v - \alpha} \Big) v \\ &= \sum_i \lambda_i \frac{a^t v - a^t s_i}{a^t v - \alpha} \bar{s}_i \ + \ \Big(\lambda_0 - \frac{\alpha \sum_i \lambda_i - \sum_i \lambda_i a^t s_i}{a^t v - \alpha} \Big) v. \end{aligned}$$

At this point we use that $x \in H$, that is, $a^t x = \sum_i \lambda_i a^t s_i + \lambda_0 a^t v = \alpha$, and that this was a convex combination, so $\sum_i \lambda_i = 1 - \lambda_0$, to conclude that the last term in large parentheses is 0. $\square$

Exercise 2.31. Let $P := \operatorname{conv}\{\pi(\pm 1, \pm 1, 0, 0) : \pi \in \mathfrak{S}_4\}$ be the convex hull of all the vectors that have two ± 1 entries and two zero coordinates.

- How many vectors are these?
- Why are they all vertices?
- Why do they all have the same vertex figure?
- Compute one vertex figure.

Proposition 2.32. *For any vertex v of a d-polytope P, the k-dimensional faces of P/v are in an inclusion-preserving bijection with the $(k+1)$-dimensional faces of P that contain v.*
In particular, P/v is a polytope of dimension $d-1$.

Proof. Clearly if F is a face of P, then $F \cap H$ is a face of $P \cap H = P/v$.
Note that $v \notin H$. Thus every $(k+1)$-face $F \subseteq P$ with $v \in P$ defines a k-face F/v of P/v: From the previous proof we can see that $\operatorname{aff}((F \cap H) \cup \{v\}) = \operatorname{aff}(F)$.
For the converse, let $G \subseteq P/v$ be a k-face, defined by the inequality $b^t x \leq \beta$. Then we calculate that this inequality, plus a suitable (not necessarily positive!) multiple of the equation $a^t x = \alpha$ defining H, is satisfied with equality on $P \cap (\operatorname{aff}(G \cup \{v\}))$, but strictly on all other points of P. Explicitly, the inequality we consider is

$$(b^t + \mu a^t)x \leq \beta + \mu\alpha, \tag{5}$$

and this will be satisfied with equality on v if $(b^t + \mu a^t)v = \beta + \mu\alpha$, that is, if $\mu = -\frac{b^t v - \beta}{a^t v - \alpha}$, where the denominator is positive. This inequality (5) is valid on P/v and valid with equality on v. Let $P = \operatorname{conv}(S \cup \{v\}$. Then the inequality is valid on all points of S as well, since a point $s \in S$ that violates it would give rise to $\bar{s} \in P/v$ that violates it as well.
Thus

$$\widehat{G} := P \cap (\operatorname{aff}(G \cup \{v\}))$$

is the desired $(k+1)$-face of P. □

End of class on October 29

2.2.2 Order theory and the face lattice

Definition 2.33 (Posets and lattices)**.** A *poset* is a partially ordered set, that is, a set S with a binary relation "$\leq$" that is *reflexive* ($x \leq x$ for all $x \in S$), *asymmetric* ($x \leq y \leq x$ implies $x = y$) and *transitive* ($x \leq y \leq z$ implies $x \leq z$). (All posets we consider are finite.) Formally, the poset could be written $(S, \leq)$, but it is customary to write the same letter S for the poset.
An *interval* in a poset $(S, \leq)$ is a subposet (i.e., a subset with the induced partial order) of the form

$$[x, y] := (\{z \in S : x \leq z \leq y\}, \leq)$$

for $x, y \in S$, $x \leq y$.
A *chain* in a poset is a totally-ordered subset.
A poset is *bounded* if it has a unique minimal element, denoted $\hat{0}$, and a unique maximal element, denoted $\hat{1}$.
A poset is *graded* if it has a unique minimal element $\hat{0}$, and if for every element x of the poset, all maximal chains from $\hat{0}$ to x have the same length, called the *rank* of the element, usually

denoted $r(x)$. The function $r : S \to \mathbb{N}_0$ is then called the *rank function* of S. If a poset is graded and has a maximal element $\hat{1}$, we write $r(S) := r(\hat{1})$ for the *rank of the poset*.

A poset is a *lattice* if any two elements a, b have a unique minimal upper bound, denoted $a \vee b$, called the *join* of a and b, and a unique maximal lower bound, denoted $a \wedge b$, and called the *meet* of a and b.

Exercise 2.34. Let $(Q, \leq)$ be a finite partial order. Show that any two of the following properties yield the third:

1. The poset is bounded.
2. Meets exist.
3. Joins exist.

Exercise 2.35. Let Q be a finite lattice, and A be an arbitrary subset. Then A has a unique minimal upper bound, the *join* $\bigvee A$, and a unique maximal lower bound, the *meet* $\bigwedge A$.

Theorem 2.36 (The polytope face lattice)**.** *The face poset* $(\mathcal{F}, \subset)$ *of any polytope is a finite graded lattice, denoted* $L = L(P)$*, of rank* $r(L(P)) = \dim(P) + 1$*.*

Proof. This is a finite bounded poset, with minimal element $\hat{0} = \emptyset$ and maximal element $\hat{1} = P$. Meet exists, as clearly $F \wedge F' = F \cap F'$ is the largest face contained in both F and F'. (The intersection is a face by Lemma 2.28.) Thus $L(P)$ is a lattice.

If $G \subset F$ are faces, then in particular G is a face of F, and thus $\dim(G) < \dim(F)$. Thus all we have to prove is that if $\dim(F) \geq \dim(G) + 2$, then there is a face H with $G \subset H \subset F$.

If $F \subset P$, then $\dim(F) < \dim(P)$, so we are done by induction.

If $\emptyset \subset G$, then G has a vertex v, and $[G, F] \subseteq [v, P] = L(P/v)$, where $\dim(P/v) < \dim(P)$, so we are done by induction.

If $G = \emptyset$ and $F = P$, where $\dim(P) \geq 1$, then P has a vertex w, where $\emptyset \subset \{w\} \subset P$. $\square$

Definition 2.37 (Combinatorially equivalent)**.** Two polytopes P and P' are *combinatorially equivalent* if their face lattices L, L' are isomorphic as posets, that is, if there is a bijection $f : L \to L'$ such that $x \leq_L y$ holds in P if and only if $f(x) \leq_{L'} f(y)$ holds in P'.

Exercise 2.38. Define "isomorphic" for posets, and for lattices. Show that if Q is a poset and L is a lattice, and if Q and L are isomorphic as posets, then Q is a lattice, and Q and L are also isomorphic as lattices.

Exercise 2.39. Let us consider the poset $D(n)$ of all divisors of the natural number n (examples to try: 24 and 42 and 64), ordered by divisibility. Are these posets? Are they bounded? Are they lattices? Graded? What is the rank function? Can you describe join and meet?

For which n is there a polytope with $D(n)$ isomorphic to its face lattice?

Lemma 2.40. *If two polytopes* P, P' *are affinely isomorphic (that is, if there is an affine bijective map* $P \to P'$*), then they are combinatorially equivalent. The converse is wrong.*

Lemma 2.41 (Face lattice of a simplex)**.** *Let* Δ_{k-1} *be a* $(k-1)$*-dimensional simplex (with* k *vertices). Its face lattice is isomorphic to the poset of all subsets of a* k*-element set, ordered by inclusion known as the* Boolean algebra B_k *of rank* k*, as given for example by* $(2^{[k]}, \subseteq)$*, where* $2^{[k]}$ *denotes the collection of all subsets of* $[k] := \{1, 2, \ldots, k\}$*.*

Proof. Any two $(k-1)$-simplices are affinely equivalent.
Any subset of the vertex set of a simplex defines a face, which is a simplex. □

Exercise 2.42. Prove that if any subset of vertex set of a polytope defines a face, then the polytope is a simplex.

Theorem 2.43 (Intervals in polytope face lattices)**.** *Let $G \subseteq F$ be faces of a polytope P. Then the interval*

$$[G, F] = (\{H \in L(P) : G \subseteq H \subseteq F\}, \subseteq)$$

of $L(P)$ is the face lattice of a polytope of dimension $\dim(F) - \dim(G) - 1$.
In particular, if $G = \emptyset$, then $[G, F] = L(F)$.
In particular, if $F = P$ and $G = \{v\}$ is a vertex, then $[G, F] = L(P/v)$.

Proof. The two "in particular" statements follow from Propositions 2.26 and 2.32. Now we can use induction. □

Corollary 2.44 (Diamond property)**.** *Any interval $[x, y]$ of length 2 in a polytope face lattice contains exactly two elements z with $x < z < y$.*

This "harmless lemma" has substantial consequences.

Corollary 2.45. *For every polytope, every face is the minimal face containing a certain set of vertices. (More precisely, every face is the convex hull of the vertices it contains.)*
Simultaneously, every face is an intersection of facets (it is the intersection of the facets it is contained in).

Proof. This says that every element in the face lattice of a polytope is a join of vertices, and a meet of facets.
This can be phrased and proved entirely in lattice-theoretic language: Take a graded lattice of rank $d + 1$ with the diamond property. Then every element of rank $r(x) \leq d$ is a meet of elements of rank $d - 1$ (which would be called "co-atoms"). Simultaneously, every element of rank $r(x) > 0$ is a join of elements of rank 1 (which are called "atoms").
To prove this, note that for an element of rank $k \geq 2$ the diamond property shows that it is the join of two elements of rank $k - 1$, and by induction those are joins of atoms. Dually for meets of coatoms. □

End of class on October 30

2.2.3 Simple and simplicial polytopes

Definition 2.46. A polytope is *simplicial* if all its facets are simplices.
A polytope is *simple* if all its vertex figures are simplices.

Lemma 2.47. *A polytope is simplicial if all the proper lower intervals in its face lattice are boolean.*
A polytope is simple if all the proper upper intervals in its face lattice are boolean.

Thus, in particular, to be simplicial or simple is a "combinatorial" property: It can be told from the face lattice.
Note that if the set of $n > d$ points $V \subset \mathbb{R}^d$ is "in general position" in the sense that no $d + 1$ points lie on a hyperplane, then $P = \mathrm{conv}(S)$ is a simplicial d-polytope.

Exercise 2.48. Every polytope that is both simple and simplicial is a simplex, or it has dimension 2.

2.2.4 $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes: The representation theorem

Theorem 2.49 (Minkowski–Weyl Representation Theorem)**.** *Every d-dimensional polytope in $\mathbb{R}^d$ can be represented in the following equivalent ways:*

$\mathcal{V}$-polytope *The subset P is given as a convex hull of a finite set $V \subset \mathbb{R}^d$:*

$$P = \mathrm{conv}(V).$$

This representation is unique if V is the set of all vertices of P.

$\mathcal{H}$-polytope *The subset P is given as the set of solutions of a finite system of linear inequalities,*

$$P = \{x \in \mathbb{R}^d : Ax \leq a\}.$$

This representation is unique if the system "$Ax \leq a$" consists of one facet-defining linear inequality for each facet of P. (Uniqueness up to permutation of the inequalities, and up to taking positive multiples of the facet-defining inequalities.)

Proof. A $\mathcal{V}$-polytope is a special representation of what we have up to now called simply a polytope. The uniqueness was proven in Proposition 2.25.
Every $\mathcal{V}$-polytope is an $\mathcal{H}$-polytope:
The fact that every $\mathcal{V}$-polytope is the solution of a finite set of inequalities follows from a procedure called "Fourier–Motzkin elimination". For this let $V = (v_1, \dots, v_n) \in \mathbb{R}^{d \times n}$. We write

$$\begin{aligned} P_{d+n} := \{ \tbinom{x}{\lambda} \in \mathbb{R}^{d+n} : \; & x = \lambda_1 v_1 + \cdots + \lambda_n v_n, \\ & \lambda_1 + \cdots + \lambda_n = 1, \\ & \lambda_1, \dots, \lambda_n \geq 0 \} \end{aligned}$$

This $P_{d+n} \subset \mathbb{R}^{d+n}$ is clearly an $\mathcal{H}$-polytope (a bounded solution of a linear system of inequalities); indeed, it is an $(n-1)$-dimensional simplex, with vertices $\tbinom{v_i}{e_i}$. Furthermore, projection of P_{d+n} to $\mathbb{R}^d$ by "deleting the last n coordinates" yields P. Thus we simply have to show that "deleting the last coordinate" maps an $\mathcal{H}$-polytope to an $\mathcal{H}$-polytope.
For this, let $\pi : P' \to P''$, $\tbinom{x}{y} \mapsto x$ be such a projection map $x \in \mathbb{R}^m$, $y \in \mathbb{R}$, where P' is given by linear inequalities (and possibly equations). A point x lies in P'' if $\tbinom{x}{y}$ lies in P' for some y. Such an y exists if all the upper bounds for y (which are given by linear inequalites in the other coordinates) are larger or equal than all the lower bounds for y (which are given similarly). Thus

"all upper bounds on y are larger or equal all lower bounds"

yields a new system of inequalities that defines P''. (If there are equations fixing y, then those have to be taken in account as well, and have to be compatible with the inequalities.)

We leave the proof of the uniqueness part for later.

End of class on November 5

Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron:

For this we prove a similar statement for more general sets: Every subset $Q \subset \mathbb{R}^d$ that is given in the form

$$Q = \{x \in \mathbb{R}^d : Ax \le a\},$$

for some $A \in \mathbb{R}^{n\times d}$ and $a \in \mathbb{R}^n$, which we call an *$\mathcal{H}$-polyhedron* (not necessarily bounded!) can be written as a *$\mathcal{V}$-polyhedron*, in the form

$$Q = \operatorname{conv}(V) + \operatorname{cone}(Y),$$

where

$$\operatorname{cone}(Y) = \{\mu_1 y_1 + \cdots + \mu_m y_m : \mu_1, \ldots, \mu_m \ge 0\}$$

is a conical combination of the vectors in the finite set $Y = \{y_1, \ldots, y_m\} \subset \mathbb{R}^d$.

To prove this, we interpret the set Q as given above as the $\mathcal{H}$-polyhedron

$$\widehat{Q} = \{\tbinom{x}{z} \in \mathbb{R}^{d+n} : Ax \le z\},$$

intersected with the subspace $\{\binom{x}{z} \in \mathbb{R}^{d+n} : z = a\}$.

This $\widehat{Q}$ we write as a $\mathcal{V}$-polyhedron: It is the sum of the linear subspace $\{\binom{x}{z} \in \mathbb{R}^{d+n} : Ax = z\}$, which has a cone basis given by the vectors $\pm\binom{e_i}{a_i}$, and an orthant $\{\binom{x}{z} \in \mathbb{R}^{d+n} : x = 0, z \ge 0\}$ spanned as a cone by unit vectors $\binom{0}{e_j}$.

So it suffices to show that the intersection of any $\mathcal{V}$-polyhedron $\widehat{Q}$ with a hyperplane of the form $H_j := \{\binom{x}{z} \in \mathbb{R}^{d+n} : z_j = a_j\}$ is again a $\mathcal{V}$-polyhedron. So let's consider

$$\widehat{Q} = \operatorname{conv}(W) + \operatorname{cone}(U) = \operatorname{conv}(W^+ \cup W^- \cup W^0) + \operatorname{cone}(U^+ \cup U^- \cup U^0),$$

where we have split the set W into the subsets lying above, on, or below the hyperplane H, and similarly with U with the hyperplane $H_j^0 := \{\binom{x}{z} \in \mathbb{R}^{d+n} : z_j = 0\}$.

In this case we get lots of points in $\widehat{Q} \cap H_j$:

- points in W^0,
- intersections of H_j with segments between a point in W^+ and one in W^-,
- intersections of H_j with rays starting from a point in W^+ with direction in U^-, and
- intersections of H_j with rays starting from a point in W^- with direction in U^+.

Let $V_j(\widehat{Q})$ be the set of all these points. Similarly, collect the following directions in $\widehat{Q} \cap H_j^0$:

- directions in U^0, and
- directions obtained by a suitable combination of a direction in U^+ and one in U^-.

Let $R_j(\widehat{Q})$ be the set of all these directions. Then it is clear that

$$\widehat{Q} \cap H_j \supset \operatorname{cone}(V_j(\widehat{Q})) + \operatorname{cone}(R_j(\widehat{Q})).$$

To prove that the converse inclusion "$\subseteq$" holds, we have to take any point $x \in \widehat{Q} \cap H_j$ and split it into contributions coming from the points and rays we have collected. It turns out that this is equivalent to finding a point in a given transportation polytope – a problem that you will solve for Problem Set 4. (Details for the computation omitted here. Example done in class.)

Every $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope:

Thus we have seen that any intersection of an $\mathcal{H}$-polyhedron with a coordinate subspace is also a $\mathcal{V}$-polyhedron, of the form $\mathrm{conv}(V) + \mathrm{cone}(Y)$. If the intersection is bounded, then clearly the $\mathcal{V}$-polyhedron is of the form $\mathrm{conv}(V)$, i.e., a $\mathcal{V}$-polytope. □

Remark 2.50. Fourier–Motzin elimination is constructive, and not hard to implement. It is contained in software systems such as `PORTA` and `polymake`.

In particular, instead of solving for upper bounds and lower bounds in a variable we want to eliminate, we just take two inequalities $a^t x \leq \alpha$ and $b^t x \leq \beta$ where for some variable x_i the coefficient in one is positive and in the other is negative, say $a_i > 0$ and $b_i < 0$. Then the positive combination of the two inequalities

$$[(-b_i)a^t + (a_i)b^t]^t x \leq (-b_i)\alpha + (a_i)\beta$$

is also valid, and it does not involve the variable x_i any more: This is the elimination step performed by adding/combining inequalities.

However, the elimination algorithm is also badly exponential: If we are "unlucky", every step transforms a system of n inequalities into $(\frac{n}{2})^2$ inequalities. So within a few steps the number of inequalities can "explode". The result will typically contain many redundant inequalities, but these are not easy to detect.

2.2.5 Polarity/Duality

Definition 2.51. Let $K \subset \mathbb{R}^d$ be a subset. Its *polar* is

$$K^* = \{y \in \mathbb{R}^d : y^t x \leq 1 \text{ for all } x \in K\}.$$

Exercise. $K^* = \mathrm{conv}(K)^* = \mathrm{conv}(K \cup \{0\})^*$.

Exercise. Compute and draw K^* for axis parallel rectangles in the plane with opposite vertices

(i) $(0, 0)$ and $(M, 1)$, for $M > 0$ large.
(ii) $(-\varepsilon, -\varepsilon)$ and $(M, 1)$, for $M > 0$ large, $\varepsilon > 0$ small.
(iii) $(\varepsilon, \varepsilon)$ and $(M, 1)$, for $M > 0$ large, $\varepsilon > 0$ small.

What happens for $\varepsilon \to 0$, $M \to \infty$?

Lemma 2.52. *Let $K, L \subseteq \mathbb{R}^d$ be a closed convex set.*

(i) $0 \in K^*$.
(ii) *K^* is closed and convex.*
(iii) *$K \subseteq L$ implies $K^* \supseteq L^*$.*
(iv) *If $0 \in K$, then $K^{**} = K$.*
(v) *If $0 \in K, L$, then $K \subseteq L$ if and only if $K^* \supseteq L^*$.*
(vi) *K is bounded if and only if K^* has 0 in its interior.*
(vii) *K^* is bounded if and only if K has 0 in its interior.*

Proof. Items (i), (ii) and (iii) are easy to see/calculate.

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For (iv), we have $K \subseteq K^{**}$ by definition. If $z \notin K$, then as K is closed and convex, by the Separation Theorem there are a vector $y \neq 0$ and $\gamma \in \mathbb{R}$ such that $y^t x < \gamma$ holds for all $x \in K$, but not for $x = z$, that is, such that $y^t z > \gamma$. As $0 \in K$, we get $\gamma > 0$, and after possibly rescaling we may assume $\gamma = 1$. Thus we have that (1) $y^t x < 1$ holds for all $x \in K$, but (2) $y^t z > \gamma$. But the first condition says that $y \in K^*$, and thus the second one says that $z \notin K^{**}$. In other words, we have proved that $K^{**} \subseteq K$.

(iii) and (iv) together yield (v).

Also (iv) immediately implies (vi) and (vii), as K is bounded if and only if $K \subseteq B(0, R)$, where $B(0, R)$ is the ball with center 0 and radius R, for some suitably large R, and similarly K has 0 in the interior if and only if $B(0, \varepsilon)$ for a suitably small $\varepsilon > 0$. □

Interestingly enough, we get a very explicit description of the polar of a polytope — assuming that we have both a $\mathcal{V}$- and an $\mathcal{H}$-representation available.

Theorem 2.53 (Polarity for polytopes)**.** *Let P be a d-polytope in $\mathbb{R}^d$ with 0 in its interior, with*

$$P = \operatorname{conv}(V) = \{x \in \mathbb{R}^d : Ax \leq 1\}$$

with $V \in \mathbb{R}^{d \times n}$ and $A \in \mathbb{R}^{m \times d}$, that is, a convex hull of n points resp. the solution set of m inequalities.

Then the polar P^* is also a d-polytope with 0 in its interior, and

$$P^* = \operatorname{conv}(A^t) = \{y \in \mathbb{R}^d : V^t y \leq 1\}.$$

Under this correspondence, the vertices of P correspond to the facets of P^*, and vice versa. In particular, if the set V was chosen minimal (that is, the set of vertices of P) and the system "$Ax \leq 1$" was minimal, then $Ax \leq 1$ consists of exactly one facet-defining inequality for each facet of P.

Proof (Part I). For this, read "$P = \operatorname{conv}(V)$" as saying that P is the convex hull of the columns of V. At the same time, "$P = \{x \in \mathbb{R}^d : Ax \leq 1\}$" says that P is the polar of the set of columns of A^t. With this, everything follows from K^{**}, if we note that the first representation yields that P is bounded, and the second one implies that 0 is in the interior. □

Exercise 2.54. For

$$P = \operatorname{conv}(V) = \{x \in \mathbb{R}^d : Ax \leq 1\} \qquad \text{and} \qquad P^* = \operatorname{conv}(A^t) = \{y \in \mathbb{R}^d : V^t y \leq 1\},$$

describe all the faces of P^* in terms of the faces of P — that is, give the $\mathcal{H}$-description of a face $F^\diamond$ of P^* in terms of the $\mathcal{V}$-description of P and F, etc.

Theorem 2.55 (Duality for polytopes)**.** *Let P be a d-polytope in $\mathbb{R}^d$ wit 0 in the interior and let P^* be its polar, then the face lattice $L(P^*)$ is the "opposite" of $L(P)$.*

Proof. There are two ways to prove this. The "hard way" is to go via Exercise 2.54, and to describe a precise match between the faces $F \subset P$ and "corresponding" faces $F^\diamond \subseteq P*$.

The easier way goes via the following observation, which plainly says that the incidences between the vertices and the facets of a polytope already fix the combinatorial type (i.e., the face lattice). □

Terminology: If $L(Q) = L(P)^{opp}$, then we say that Q is *a dual* of P. Note that every polytope has many duals, but only one polar polytope (if it has 0 in the interior etc.)

Corollary 2.56. *A polytope is P is simple if and only if P^* is simplicial, and vice versa.*

Theorem 2.57. *Let P be a d-dimensional polytope with n vertices and m facets. Then the combinatorial type of P (that is, the face lattice $L(P)$) is determined by the vertex–facet incidences, that is, by the matrix*

$$I(P) = (\kappa_{ij}) \in \{0,1\}^{n\times m},$$

where $\kappa_{ij} = 1$ if $v_i \in F_j$, and $\kappa_{ij} = 0$ otherwise, for some arbitrary labelling $v_1, \dots, v_n$ of the vertices and $F_1, \dots, F_m$ of the facets.

Proof. The faces are the intersections of facets, and the vertex sets of faces are exactly the intersections of vertex sets of facets, by Corollaries 2.27 and 2.45.
Thus the vertex sets of facets are given by the rows of the matrix $I(P)$, and the vertex sets of faces are exactly the intersections of these rows, which we interpret as incidence vectors of vertex sets of facets. □

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Lemma 2.58 (Characterization of vertices)**.** *Let $P = \operatorname{conv}(V) = \{x \in \mathbb{R}^d : Ax \le 1\}$. Then $v_0 \in \mathbb{R}^d$ is a vertex of P if and only if any one of the following conditions are satisfied:*

(i) *$\{v_0\}$ is a face of dimension 0, that is, $v_0 \in P$ but there is an inequality $a^t x \le \alpha$ such that $a^t v_0 = \alpha$, while $a^t v_i < \alpha$ for all other $v_i \in V$.*
(ii) *$v_0 \in V$, and there is an inequality $a^t x \le 1$ such that $a^t v_0 = 1$, while $a^t v_i < 1$ for all other $v_i \in V$.*
(iii) *v_0 is a point in P such that $\{v_0\}$ is an intersection of some facets of P.*
(iv) *v_0 is a point in P such that $\{v_0\}$ is an intersection of d facet-defining hyperplanes $H_i = \{x \in \mathbb{R}^d : a_i^t x = 1\}$ $(1 \le i \le d)$.*

Proof. (i) is the definition of a vertex (0-dimensional face).
(ii): As 0 lies in the interior of P, the inequality from (i) has to have $\alpha > 0$, so we can rescale to get $\alpha = 1$. Also v_0 lies in V, otherwise we would have $a^t v < 1$ for all $v \in V$ and thus $a^t x < 1$ for all $x \in P$.
(iii): We know that every face (and thus every vertex) is an intersection of facets. Conversely, every intersection of facets is a face, and if the face is a single point, it is a vertex.
(iv): If v_0 is a vertex, then it is contained in a maximal chain of faces $v_0 = G_0 \subset G_1 \subset G_{d-2} \subset G_{d-1}$, where G_i is a face of dimension i and $G_i = G_{i+1} \cap F_i$, where F_i is a facet — since every face is an intersection of facets. Let $H_i = \operatorname{aff}(F_i)$, then we have that $F_i \subset H_i$ and $F_i = P \cap H_i$, and thus

$$G_i = G_{i+1} \cap F_i \subseteq G_{i+1} \cap H_i \subseteq G_{i+1} \cap P \cap H_i = G_{i+1} \cap F_i,$$

which yields $G_i = G_{i+1} \cap H_i$. We conclude that each of the intersections H_{d-1}, $H_{d-1} \cap H_{d-2}$, $H_{d-1} \cap H_{d-2} \cap \cdots \cap H_0$ strictly contains the next one — and thus the last one in the sequence has dimension 0, it is a single point, namely $G_0 = \{v_0\}$. On the other hand, v_0 is then an intersection of facets, so it is a vertex. □

Lemma 2.59 (Characterization of facets). *Let* $P = \mathrm{conv}(V) = \{x \in \mathbb{R}^d : Ax \leq 1\}$. *Then* $F \subset P$ *is a facet of* P *if and only if any one of the following conditions are satisfied:*

(i) $F = \{x \in P : a^t x = \alpha\}$ *for an inequality* $a^t x \leq \alpha$ *that is valid for all of* P*, with* $\dim(F) = d - 1$.

(ii) $F = \{x \in P : a^t x = \alpha\}$ *for an inequality* $a^t x \leq \alpha$ *that is valid for all of* P*, with* d *affinely-independent points* $v_1, \dots, v_d$ *from the set* V *that satisfy* $a^t v_i = 1$.

(iii) $F = \{x \in P : a_i^t x = 1\}$ *for an inequality* $a_i^t x \leq 1$ *from the system* $Ax \leq 1$*, with* d *affinely-independent points* $v_1, \dots, v_d$ *from the set* V *that satisfy* $a_i^t v_j = 1$.

Proof. (i) is the definition of a facet.

(ii): Let $V_0 \subseteq V$ be the subset of all the $v_i \in V$ that satisfy the inequality from (i) with equality. If the affine hull of these points has dimension $d-1$, then we can choose d that span this hull. If the affine hull has smaller dimension, then we note $F \subset \mathrm{aff}(V_0)$, so F is not a facet.

(iii): Here the new information is that the facet-defining inequalities all come from the system $Ax \leq 1$. However, note that the inequality $a^t x \leq 1$ that satisfies $a^t v_j = 1$ for $1 \leq j \leq d$ is unique. If it were not in the inequality system, then the barycenter $\frac{1}{d}(v_1^+ \cdots + v_d)$ would lie in the *interior* of the set defined by $Ax \leq 1$; on the other hand, it lies on the boundary due to the inequality $a^t x \leq 1$. □

Proof of Theorem 2.53 (Part II). From the characterization Lemma 2.59, we see that the facets of P are exactly given by the inequalities of the system $Ax \leq 1$, under the assumption that the system was chosen to be minimal.

The assumption that the two systems for P are minimal implies that the systems for P^* are also minimal, otherwise we would get a contradiction to $P^{**} = P$. □

Proposition 2.60. *The incidence matrix* $I(P)$ *may be a rather compact encoding of a polytope, but it is not so easy to read things off.*

(1) *To get the dimension* d *of a polytope from* $I(P)$ *we have to find a sequence of columns such that the first column is arbitrary (corresponding to a facet) and each subsequent one is chosen to have a* maximal *intersection with the intersection of the previously-chosen ones, thus yielding the next face of a maximal chain.*

(2) *The incidence matrix of the polar is the transpose of the matrix:* $I(P^*) = I(P)^t$.

(3) *If* $\dim(P) = d$*, then* P *is simplicial if each column of* $I(P)$ *contains exactly* d *ones.*

(4) *If* $\dim(P) = d$*, then* P *is simple if each row of* $I(P)$ *contains exactly* d *ones.*

Proof of Theorem 2.55. With completing the proof of Theorem 2.53, we get that the vertices of P correspond to the facets of P^*, and vice versa. Thus the $I(P^*)$ is the transpose of $I(P)$, and thus $L(P^*)$ is the opposite of $L(P)$. □

2.2.6 The Farkas lemmas

"The Farkas lemma" is a result that comes in many different flavors; it says that if something happens in polyhedral combinatorics, then there always is a concrete reason. Here is a basic version:

Proposition 2.61. *A system* $Ax \leq a$ *has no solution if and only if there is a vector* $c \geq 0$ *such that* $c^t A = 0$ *and* $c^t a = -1$.

Proof. The Farkas lemmas can be derived from Separation Theorems, or from each other, or from Fourier–Motzkin. We sketch Fourier–Motzkin: We can eliminate *all* the variables from the system $Ax \leq a$, such that the resulting system of inequalities $0 \leq \gamma_i$ has a solution if and only if the original system has none. Moreover, all inequalities in the resulting system are non-negative combinations of the inequalities in the original system.

Thus if the resulting system has no solution, then one inequality ready "$0 \leq \gamma_i$" for some $\gamma_i < 0$. Indeed, we may rescale to get $\gamma_i = -1$. The inequality is obtained by non-negative combination, that is, $c^t A = 0$, $c^t a = \gamma_i = -1$.

Conversely, check that the existence of c implies that the system has no solution. □

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3 Polytope theory

3.1 Examples, examples, examples

What do we want to know?

- dimension d
- number of vertices $f_0 = n$, number of facets f_{d-1}
- $\mathcal{V}$- and $\mathcal{H}$-description
- f-vector $(f_0, f_1, \ldots, f_{d-1})$
- graph
- simple? simplicial?
- [diameter, surface area, volume? – not so much a topic of this course]
- dual polytope?
- symmetries?
- combinatorial type? incidence matrix?
- face lattice L
- etc.

We will mix a discussion of specific (classes of) examples with a discussion of constructions – which produce new examples.

Note that the various classes of examples we describe will not be disjoint (example: every simplex is a pyramid, every cube is a prism, a triangle is both a simplex and a polygon, etc.)

3.1.1 Basic building blocks

Example 3.1 (The (regular) convex polygons). Let P be any 2-dimensional polytope, and $n = f_0(P)$ its number of vertices. Then $n \geq 3$ and $f_1(P) = n$. Any two 2-polytopes with n vertices are combinatorially equivalent — and they are in particular equivalent to the *regular convex n-gon* given by

$$P_2(n) = \operatorname{conv}\{(\cos(\tfrac{k}{n}2\pi), \sin(\tfrac{k}{n}2\pi)) : 0 \leq k \leq n\}.$$

This example in particular contains the complete classification of 2-dimensional polytopes.

Example 3.2 (The d-simplex). The standard simplex of Definition 2.17 may be described as

$$\begin{aligned}\Delta_d &= \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{d+1},\ \lambda_0, \ldots, \lambda_d \geq 0,\ \lambda_1 + \cdots + \lambda_{d+1} = 1\} \\ &= \operatorname{conv}\{e_1, \ldots, e_{d+1}\}.\end{aligned}$$

This is a d-dimensional polytope in $\mathbb{R}^{d+1}$. It has $d+1$ vertices and $d+1$ facets; the k-faces correspond to the $(k+1)$-subsets of $[d+1]$. In particular, the face lattice is a boolean algebra B_{d+1}, and we get

$$f_k(\Delta_d) = \binom{d+1}{k+1}.$$

The standard d-simplex has the symmetry group $\mathfrak{S}_{d+1}$, acting by permutation of coordinates (and thus of vertices).

A full-dimensional version of the standard simplex is obtained by deleting the last coordinate,

$$\begin{aligned}\Delta'_d &= \{(\lambda_1,\dots,\lambda_d)\in\mathbb{R}^{d+1},\ \lambda_0,\dots,\lambda_d\ge 0,\ \lambda_1+\cdots+\lambda_d\le 1\}\\ &= \operatorname{conv}(0,e_1,\dots,e_d).\end{aligned}$$

This simplex has volume $\frac{1}{d}$. It has a smaller symmetry group than Δ_d. We leave it as an exercise to construct and describe a full-dimensional fully-symmetric realization of Δ_d.

Example 3.3 (The d-cube). Again there are two very familiar versions of the d-dimensional cube, the $0/1$-cube

$$C_d^{01} = \operatorname{conv}\{0,1\}^d = \{x\in\mathbb{R}^d : 0\le x_k\le d\} = [0,1]^d$$

and the ± 1-cube

$$C_d = \operatorname{conv}\{1,-1\}^d = \{x\in\mathbb{R}^d : -1\le x_k\le d\} = [-1,1]^d = \{x\in\mathbb{R}^d : \|x\|_\infty\le 1\}.$$

They are equivalent by a similarity transformation.

The non-empty k-faces are obtained by choosing k coordinates which have the full range of $[0,1]$ resp. $[-1,1]$ and fixing the other $d-k$ vertices to the lower or upper bound. In particular, this yields

$$f_k(C_d) = 2^{d-k}\binom{d}{k}$$

for $k\ge 0$, while $f_{-1}=1$. In particular, the d-cube has 3^d non-empty faces.

The d-cube is simple.

The symmetry group of C_d is generated by the permutations of coordinates and by the reflections in coordinate hyperplanes. It has $2^d d!$ elements, and is known as the group of signed permutations, or as the *hyperoctahedral group*.

Exercise 3.4. For which $k = k(d)$ does the d-cube have the largest number of k-faces? To answer this, analyze the quotients f_k/f_{k-1}, and show that they decrease with k. Conclude that the f-vector of the d-cube is *unimodal*, that is,

$$f_0 < f_1 < \cdots < f_{k(d)} \ge f_{k(d)+1} > \cdots > f_{d-1}.$$

Example 3.5 (The d-dimensional crosspolytope[2]). The standard coordinates for the *d-dimensional crosspolytope* are given by

$$\begin{aligned}C_d^* &= \operatorname{conv}\{\pm e_1,\dots,\pm e_d\}^d\\ &= \{x\in\mathbb{R}^d : \pm x_1+\cdots+\pm x_d\le 1\} = \{x\in\mathbb{R}^d : \|x\|_1\le 1\}.\end{aligned}$$

The proper k-faces are obtained by choosing $k+1$ coordinates, and a sign for each of them, so

$$f_k(C_d^*) = 2^{k+1}\binom{d}{k+1}$$

for $k<d$, while $f_d=1$. In particular, the d-crosspolytope has 3^d non-empty faces. And indeed, this is the polar dual of the d-cube, so in particular it has the same number of faces. The d-crosspolytope is simplicial. Its symmetry group is again the hyperoctahedral group.

[2] Compare Problem Sheet 2 (Problem 2).

Exercise 3.6 (The Half-cube)**.** Let

$$H_d := \text{conv}\{x \in \{0,1\}^d : x_1 + \cdots + x_d \in 2\mathbb{Z}\}$$

be the dth *half-cube*.

(i) Describe H_d for $d \leq 4$.
(ii) Describe the facets of H_d: How many are they, what are their combinatorial types? (The cases $d = 1, 2, 3$ need to be argued separately.)
(iii) Give an $\mathcal{H}$-representation of H_d. (The cases $d = 1, 2, 3$ need to be argued separately.)
(iv) Show that H_d is "3-simplicial," that is, all its 3-faces are tetrahedra.
(v) Show that H_d is simplicial for $d \leq 4$, but not for $d > 4$.

3.1.2 Some basic constructions

Proposition 3.7 (Product[3])**.** *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes, then the product*

$$P \times Q \subset \mathbb{R}^{d+e}$$

is a polytope of dimension $\dim(P)+\dim(Q)$*. Its non-empty faces are the products of non-empty faces of P and of Q. Thus the product construction is combinatorial: the face lattice of $P \times Q$ can be derived from the face lattices of P and of Q. In particular,*

$$f_k(P \times Q) = \sum_{i=0}^{k} f_i(P) f_{k-i}(Q) \qquad \textit{for } k \geq 0.$$

$P \times Q$ is simple if and only if P and Q are simple.
$P \times Q$ is never simplial, unless one of P and Q is 0-dimensional, or they are both 1-dimensional (and $P \times Q$ is a quadrilateral).

Example 3.8 (Prisms). Let $P \subset \mathbb{R}^d$ be a polytope. If I is an interval (that is, a 1-dimensional polytope, such as $I = [0,1]$ or $I = [-1,1]$), then the product $P \times I \subset \mathbb{R}^{d+1}$ is a *prism* over P. Then $\dim(P \times I) = \dim(P) + 1$. The non-empty faces of the prism $P \times I$ for $I = [0,1]$ are

- the faces of the *base* $P \times \{0\}$, which is isomorphic to P,
- the faces of the *top* $P \times \{1\}$, which is also isomorphic to P,
- the *vertical* faces of the form $P \times I$, where every non-empty k-face of P corresponds to a unique vertical $(k+1)$-face of P.

This also yields a drawing of the face lattice of P.

Note: the d-cube, $d > 0$, is an iterated prism.

Exercise 3.9. Define the *f-polynomial* of a d-polytope as

$$f_P(t) := 1 + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d + t^{d+1}.$$

[3] Compare Problem Sheet 2 (Problem 1).

(a) Describe the f-polynomial $f_{P\times I}$ of the prism $P \times I$ in terms of the f-polynomial of P. Deduce from this a formula for the f-polynomial of the d-cube.
(b) Describe the f-polynomial of P^* in terms of the polynomial of P. Deduce from this a formula for the f-polynomial of the d-crosspolytope.
(c) Describe the f-polynomial of $P \times Q$ in terms of the polynomials of P and of Q.

Proposition 3.10 (Direct sum[4])**.** *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes with the origin in the interior, then*

$$P \oplus Q := \mathrm{conv}(P \times \{0\} \cup \{0\} \times Q) \subset \mathbb{R}^{d+e}$$

is a polytope of dimension $\dim(P) + \dim(Q)$.
Its proper faces are all of the form $F * G := \mathrm{conv}(F \times \{0\} \cup \{0\} \times G)$, where $F \subset P$ and $G \subset Q$ are proper faces, and $\dim(F * G) = \dim(F) + \dim(G) + 1$.
In particular the direct sum is combinatorial.

Example 3.11 (Bipyramids). If P is a polytope, then $P \oplus I$ is a *bipyramid* over P: It has dimension $\dim(P) + 1$, $f_0(P) + 2$ vertices, $2f_{\dim(P)-1}$ facets, etc.
For example, the d-crosspolytope is an (iterated) bipyramid.

Proposition 3.12. *Product and direct sum are dual constructions: If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are polytopes with the origin in the interior, then*

$$(P \times Q)^* = P^* \oplus Q^*.$$

Example 3.13 (A neighborly polytope). The direct sum $\Delta_2 \oplus \Delta_2$ [constructed from two triangles with the origin in the interior] is neighborly: This is a 4-dimensional polytope with $f_0 = 6$ vertices such that any two vertices are joined by an edge. In particular, $f_1(\Delta_2 \oplus \Delta_2) = \binom{f_0}{2} = \binom{6}{2} = 15$.

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Proposition 3.14 (Joins)**.** *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes, then the* join

$$P * Q := \mathrm{conv}\big(\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} : x \in P\big\} \cup \{ \begin{pmatrix} 0 \\ y \\ 1 \end{pmatrix} : y \in Q\}\big) \subset \mathbb{R}^{d+e+1}$$

is a polytope of dimension $\dim(P) + \dim(Q) + 1$.
Its faces are the joins of the faces of P and the faces of Q. Thus the join construction is combinatorial: the face lattice of $P \times Q$ can be derived from the face lattices of P and of Q — it is simply the product,

$$L(P * Q) \cong L(P) \times L(Q).$$

In particular,

$$f_k(P * Q) = \sum_i f_i(P) f_{k-i-1}(Q) \qquad \textit{for all } k.$$

$P * Q$ is neither simple nor simplicial, except if both P and Q are simplices, or if one them is empty and the other one is simple resp. simplicial.

[4] Compare Problem Sheet 5 (Problem 1).

Example 3.15 (Pyramids). $P * \{v\}$ is the *pyramid* over P.

Remark 3.16. We are usually interested in polytopes only up to affine transformations. Thus we perform constructions such as products, direct sums, and joins in more generality than the one indicated above. Also, often P and Q lie in the same higher-dimensional vector space, and we want to see their product/join/direct sum in the same space:

- If P, Q lie in transversal affine subspaces of a real vectorspace $V \cong \mathbb{R}^N$, e.g. $P \subset V'$, $Q \subset V''$, $V' \cap V'' = \{p\}$, then
$$\{x + y : x \in P,\ y \in Q\}$$
is (affinely equivalent to) the product of P and Q.
- If P, Q lie in transversal affine subspaces V' resp. V'' of V, where $V' \cap V''$ is a relative interior point of P and of Q, then
$$\operatorname{conv}(P \cup Q)$$
is (affinely equivalent to) the direct sum of P and Q.
- If P, Q lie in skew subspaces of V, then
$$\operatorname{conv}(P \cup Q)$$
is (affinely equivalent to) the join of P and Q.

Proof. ... left as an exercise. It helps to know the definitions, e.g. the following ... □

Definition (*Reminder* from Lemma 2.40: Affine equivalence, a.k.a. affinely isomorphic)**.** Affine maps between vector spaces V and W are maps that satisfy $f(\lambda x + (1 - \lambda)y = \lambda f(x) + (1 - \lambda)f(y)$; they have the form $f(x) = Ax + b$ for a suitable matrix A and vector b.

Two polytopes $P \subset V$ and $Q \subset W$ are *affinely equivalent* if there is an affine map $f : V \to W$ such that $f(P) = Q$, where $f : P \to Q$ is a bijection. (Note that this does not require that $f : V \to W$ is a bijection – f does not need to be injective or surjective.)

Affine equivalence is an equivalence relation. In particular, affinely equivalent polytopes are combinatorially equivalent (for this, recall Lemma 2.40.)

Exercise 3.17. Show that the join construction is self-dual,
$$(P * Q)^* \cong (P^* * Q^*).$$
How do you have to interpret/adapt the notations/constructions to make this true?

Exercise 3.18. In $\mathbb{R}^d$, what is the smallest example of a polytope that is not (combinatorially equivalent to) a join, a product or a direct sum? After you have answered that: How did you interpret "smallest"?

Example 3.19 (The Hanner polytopes/The 3^d conjecture). The *Hanner polytopes* are defined as all polytopes that can be generated from $[-1, +1]$ by repeatedly applying products, direct sums, and polarity. This includes the d-dimensional cube and the d-dimensional cross polytope, but many more polytopes. (For example, a prism over an octahedron.)

All d-dimensional Hanner polytopes are centrally symmetric, and they have exactly $3^d + 1$ faces (equivalently: 3^d non-empty faces; equivalently: 3^d proper faces).

The 3^d Conjecture (by Gil Kalai, 1988 [2]) says that every centrally-symmetric d-polytope has at least $3^d + 1$ faces, and that in the case of equality it is (equivalent to) a Hanner polytope. Up to now, this is proved only for $d \leq 4$; see Sanyal, Werner & Ziegler [3].

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3.1.3 Stacking, and stacked polytopes

Definition 3.20 (Stacking)**.** Let P be a polytope and F a facet. *Stacking* a pyramid onto a facet F yields a polytope

$$P' := \mathrm{conv}(P \cup \{v_0\}) = P \cup \mathrm{conv}(F \cup \{v_0\})$$

with a new vertex v_0 such that all such that all proper faces of P, except for F, are also facets of $\mathrm{conv}(P \cup \{v_0\})$

Lemma 3.21. *Let P be a d-polytope, and $F \subset P$ a facet.*
The proper faces of $P' := \mathrm{Stack}(P, F)$ are
– *all proper faces of P, except for F, and*
– *the pyramids $\mathrm{conv}(G \cup v_0)$, for all proper faces $G \subset F$.*
The f-vector of P' is hence

$$f_i(P') = \begin{cases} f_i(P) + f_{i-1}(F) & \textit{for } i < d-1 \\ f_{d-1}(P) + f_{d-2}(F) - 1 & \textit{for } i = d-1. \end{cases}$$

Definition 3.22 (Beneath/beyond)**.** Let $P \subset \mathbb{R}^d$ be a d-polytope, and $F \subset P$ a facet.
A point $v \notin P$ lies *beneath* the facet F if v and the interior of P lie on the same side of the hyperplane H_F spanned by F.
A point $v \notin P$ lies *beyond* the facet F if v and the interior of P lie on different sides of the hyperplane H_F spanned by F.

Thus "stacking onto a facet F" describes the situation when a new point/vertex lies beyond one particular facet $F \subset P$ and beneath *all other* facets of P.

Exercise 3.23. Let $P \subset \mathbb{R}^d$ be a d-polytope, and $F \subset P$ a facet. Let $v_1, \ldots, v_n$ be the vertices of P, and assume that $v_1, \ldots, v_m$ for some $m < n$ are the vertices of F.
Show that

$$(1-\lambda)\tfrac{1}{n}(v_1 + \cdots + v_n) + \lambda\tfrac{1}{m}(v_1 + \cdots + v_m)$$

– for $\lambda = 0$ is a point in the interior of P,
– for $\lambda = 1$ is a point in the relative interior of F,
– for $\lambda > 1$ is a point beyond F, which lies beneath all other facets of P if λ is small enough (but larger than 1).

Definition 3.24. A d-dimensional *stacked polytope* $\mathrm{Stack}_d(d+1+n)$ on $d+1+n$ vertices, for $n \geq 0$, is obtained from a d-simplex $\Delta_d \subset \mathbb{R}^d$ by repeating the operation "stacking onto a facet" n times.

Exercise 3.25. Show that for $d \geq 3$ and sufficiently large n, there are different combinatorial types of stacked d-polytopes on $d+1+n$ vertices.
Discuss how the combinatorial type of $\mathrm{Stack}_d(d+1+n)$ can be described in terms of a (graph theoretical) tree. Do different trees describe different polytopes? Do different stacked polytopes have different trees?
Use this to estimate the number of different stacked polytopes $\mathrm{Stack}_d(d+1+n)$ for some fixed $d \geq 3$ and large n.

Proposition 3.26. *The f-vector of* $\mathrm{Stack}_d(d+1+n)$ *is*

$$f_i(\mathrm{Stack}_d(d+1+n)) = \begin{cases} \binom{d+1}{i+1} + n\binom{d}{i} & \text{for } i < d-1 \\ d+1+n(d-1) & \text{for } i = d-1. \end{cases}$$

Exercise 3.27. Compute and sketch the f-vector of the stacked polytope $\mathrm{Stack}_{10}(42)$. In particular, how many facets does it have? Which is the largest entry of the f-vector?

Proposition 3.28. *The stacked polytope* $\mathrm{Stack}_3(8)$ *obtained by stacking onto all* 4 *facets of a tetrahedron cannot be realized with all vertices on a sphere, so it is not* inscribable.

Proof. Stereographic projection from a tetrahedron vertex, and then an angle count in the resulting *Delaunay triangulation*. See Gonska & Ziegler [1]. (Delaunay triangulations will be discussed later.) □

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