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Discrete Geometry II – Problem Sheet 1

Please hand in your solutions to Prof. Ziegler on **Tuesday, Apr. 29, 2014** before the lecture begins.

Problem 1: *Simpleness assumption* (6 Points)

Let $A \in \mathbb{R}^{n\times d}$ be a matrix with n rows and d columns. Let $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^d$ be vectors. Assume further that A has no zero rows and has full rank d. We consider the linear program

$$(\mathrm{P}) : \quad \left\{\begin{matrix} \max & c^t x & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{matrix}\right\}.$$

Let $P \coloneqq \{x \in \mathbb{R}^d : Ax \leq b\}$ be the polyhedron of (P). We assume that $\dim P = d$. We call (P) *simple* if P is simple. Let $\varepsilon > 0$. If $b = (b_1, \ldots, b_n)^t$, then define $b_\varepsilon \coloneqq (b_1 + \varepsilon, b_2 + \varepsilon^2, \ldots, b_n + \varepsilon^n)^t$. Show that for a suitable, very small $\varepsilon > 0$, the resulting linear program (P_ε) (obtained from (P) by replacing b by b_ε) is simple and still fulfills the same hypotheses as (P).

Problem 2: *How many vertices can P have?* (6 Points)

Let (P) be as in Problem 1 above.

Use the Upper Bound Theorem for d-polytopes to derive good upper bounds for the number of vertices of the polyhedron P. (Note that P need not be bounded!) How about lower bounds on the number of vertices of P?

Problem 3: *Example: 2-d linear interpolation* (8 (+2) Points)

Assume we are given n data points $(x_1, y_1), \ldots, (x_n, y_n)$ in the plane, which we would like to approximate by a line, that is, by an affine function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ of the form $x \longmapsto ax + b$ for $a, b \in \mathbb{R}$.

Construct a linear program to compute values a and b that minimize the "ℓ_1 error term"

$$\sum_{i=1}^{n} |ax_i + b - y_i|.$$

Construct a second linear program to compute values a and b that minimize the "ℓ_∞ error term"

$$\max_{1 \leq i \leq n} |ax_i + b - y_i|.$$

Note: The "usual" way to approxiate is the ℓ_2-approximation, which doesn't need linear programming.

Bonus: Solve the two linear programs using a computer. For an online solver, see http://www.phpsimplex.com/en/.