Berlin Mathematical School

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Discrete Geometry II – Problem Sheet 2

Please hand in your solutions to Prof. Ziegler on **Tuesday, May. 6, 2014** before the lecture begins.

Problem 1: *Transforming models* (4 Points)

Assume we are given the following model ("Model 1") of a linear program

$$(\mathrm{P}): \quad \left\{ \begin{matrix} \max & c^t x & & \\ & Ax & \leq & b \end{matrix} \right\}$$

where A has n rows and d columns.

(a) Describe a method to transform linear programs in Model 1 into the following form ("Model 2"):

$$(\mathrm{P}): \quad \left\{ \begin{matrix} By & = & f \\ y & \geq & 0 \end{matrix} \right\}.$$

How many variables and equalities do we now have?

(b) Show that every Model 2 of (P) can be transformed into the form of Model 1.

(c) If we use your transformation of Model 1 into the form of Model 2, and then transform the resulting system back into form of Model 1, how many variables and inequalities do we then have?

Problem 2: *Complexity issues* (8 Points)

Let $A \in \{0, 1, -1\}^{n \times n}$ be a $0/\pm 1$ matrix. Show that

(a) The determinant of $A \in \{0, 1\}^{n \times n}$ can be arbitrarily large, even if there are only two 1's per row.

(b) The determinant of $A \in \{0, 1, -1\}^{n \times n}$ is bounded if there is at most one 1 and at most one -1 per row.

(c) Use the Hadamard inequality to give an upper bound for $|\det A|$.

(d) Assume now that $A \in \{0, 1\}^{n \times n}$. Give a much better upper bound by

— multiplying the matrix by 2,

— adding a column of 0's and then a row of 1's,

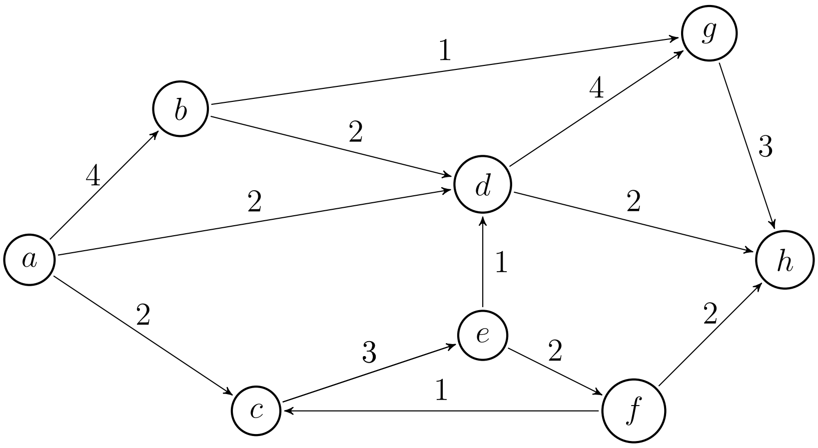
— subtracting the newly added row from all others,

and then applying the Hadamard inequality to the resulting ± 1-matrix.

(e) Give an example where the bound in (d) is tight.

Problem 3: *Network flows and integer programs* (8 Points)

Assume we are given a finite network $D = (V, A)$ with source and sink and with integer-valued arc capacities. That is, D is a finite directed graph with finitely many vertices where each arc $(x, y) \in A$ has a capacity $c(x, y) \in \mathbb{Z}_{\geq 0}$. Additionally, there are two distinguished vertices $s, t \in V$ called the *source* and the *sink*. A *flow* f through the network D is a function $f\colon V \times V \longrightarrow \mathbb{R}$ such that for all $(x, y) \in A$ we have $0 \leq f(x, y) \leq c(x, y)$. Furthermore "flow conservation" is required to hold at each vertex $v \in V \setminus \{s, t\}$, that is, the sum of the flow values on arcs into such a vertex equals the sum of the flow values on arcs leaving the vertex. Our goal is to measure the total flow, which can be calculated as the sum of the values of f on all the arcs that leave the source.



Example of a network with source "a" and sink "h" and integer capacities as indicated by the labels of the edges.

(a) Describe a linear program to determine the maximal flow for a network D as introduced above. How many rows and columns do you need — for the example given in the figure, and for a general network?
What numbers can occur as matrix entries?
What is the form of your LP from (a)? Is it one of the "models" in Problem 1?

(b) Describe and interpret the dual linear program.

(c) What type of matrix is A?
Show that the feasible basic solutions of (P) are in fact integral.