Berlin Mathematical School

Freie Universität Berlin

Prof. Günter M. Ziegler
Albert Haase, Marie Litz

Institut für Mathematik
Arbeitsgruppe Diskrete Geometrie

Discrete Geometry II – Problem Sheet 3

Please hand in your solutions to Prof. Ziegler on **Tuesday, May 13, 2014** before the lecture begins.

Problem 1: *Convex hulls* (6 Points)

(a) Give an example of a closed subset of $\mathbb{R}^2$ whose convex hull is not closed.

(b) Prove that the convex hull of an open set in $\mathbb{R}^d$ is open.

Problem 2: *Convex bodies* (8 Points)

Recall that a subset $C \subset \mathbb{R}^d$ is called a *convex body* if it is convex, compact, and full-dimensional.

(a) Determine the cardinality of the set of convex bodies in $\mathbb{R}^2$.

(b) Show that there is a sequence $K_1, K_2, \ldots$ of convex bodies in $\mathbb{R}^2$ such that for every $\varepsilon > 0$ and for each convex body $C \subset \mathbb{R}^2$, there is some i such that $K_i \subseteq C$ and $\text{vol}(K_i) \geq (1-\varepsilon)\text{vol}(C)$.[1]

Problem 3: *Extreme and exposed points* (6 Points)

Let $C \subseteq \mathbb{R}^d$ be a non-empty convex set.

(a) A point $z \in C$ is called an *extreme point* of C if for every $x, y \in C$, $z \in \text{conv}\{x, y\}$ if and only if $z = x$ or $z = y$.
Show that z is an extreme point of C if and only $C \setminus \{z\}$ is convex. Infer that if X is a finite set and $z \in \text{conv}(X)$ is an extreme point, then $z \in X$.

(b) A point $z \in C$ is called an *exposed point* of C if there is a vector $c \in \mathbb{R}^d$ such that $c^t z > c^t x$ for all $x \in C \setminus \{z\}$.
Show that every exposed point is extreme, but extreme points are not exposed in general.

(c) Let $z \in C$ be a point such that $\|z\|_p \geq \|y\|_p$ for some $1 < p < \infty$ and for all $y \in C$. Show that z is an exposed point of C.

[1] To be more precise, we could say that "vol" is the two-dimensional Lebesgue measure.