Berlin Mathematical School

Freie Universität Berlin

Prof. Günter M. Ziegler
Albert Haase, Miriam Schlöter

Institut für Mathematik
Arbeitsgruppe Diskrete Geometrie

# Discrete Geometry II – Problem Sheet 4

Please hand in your solutions to Prof. Ziegler on **Tuesday, May 20, 2014** before the lecture begins.

## Problem 1: *The relative interior of a convex set is convex* (2+2+2 Points)

Let us prove the first part of Proposition 3.11 from the lecture, which asserts that for every convex set $K$ the relative interior $\operatorname{relint}(K)$ is also convex. Consider two points $x_0$ and $x_1$ in $\operatorname{relint}(K)$ together with epsilon balls $B_{\varepsilon_0}(x_0)$ and $B_{\varepsilon_1}(x_1)$ contained in $\operatorname{relint}(K)$. Let $x = \lambda x_0 + (1-\lambda)x_1$ for $\lambda \in (0,1)$ and define $\varepsilon := \lambda\varepsilon_0 + (1-\lambda)\varepsilon_1$.

(a) Show that $B_\varepsilon(x) \subset \operatorname{conv}\big(B_{\varepsilon_0}(x_0) \cup B_{\varepsilon_1}(x_1)\big)$. This implies that $x \in \operatorname{relint}(K)$.

(b) Show that the choice of $\varepsilon$ is maximal in order for the claim in $(a)$ to hold.

(c) Show that if $x_0 \in \operatorname{relint}(K)$ and $x_2 \in K$, the point $\lambda x_0 + (1-\lambda)x_2 \in \operatorname{relint}(K)$ for $0 < \lambda \le 1$.

## Problem 2: *Intersections of convex sets* (6 Points)

Let $n \ge d+1$ and let $A_1, A_2, \dots, A_n$ and $C$ be convex subsets of $\mathbb{R}^d$. Assume that for any $(d+1)$-element index set $I \subseteq \{1, \dots, n\}$ there is a $t_I \in \mathbb{R}^d$ such that

$$t_I + C \subseteq \bigcap_{i \in I} A_i.$$

Show that there is a $t_0 \in \mathbb{R}^d$ such that $t_0 + C \subseteq A_i$ for all $i = 1, \dots, n$.
*Hint:* Do induction on $n$ and apply Radon's Theorem.

## Problem 3: *Carathéodory's Theorem in a special case* (8(+2) Points)

A topological space $X \subseteq \mathbb{R}^d$ is *path-connected* if any two points $x_0, x_1 \in X$ can be connected by a continuous path, that is, a continuous map $\gamma\colon [0,1] \longrightarrow X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

(a) Let $A$ be a convex body in $\mathbb{R}^d$. Show that if the set of extreme points $\operatorname{ext}(A)$ of $A$ is path-connected, then every point in $A$ lies in the convex hull of $d$ points from $\operatorname{ext}(A)$. This improves the bound of $d+1$ points in Carathéodory's Theorem by 1. *Note:* This is not an easy exercise. Do the best you can while still being explicit in your proof.

(b) *Bonus:* Is (a) still true if we weaken the assumption that $\operatorname{ext}(A)$ is path-connected and ask that $\operatorname{ext}(A)$ only be connected (the other assumptions remain unchanged)?