Berlin Mathematical School

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Discrete Geometry II – Problem Sheet 5

Please hand in your solutions to Prof. Ziegler on **Tuesday, May 27, 2014** before the lecture begins.

Problem 1: *Nearest and extreme points* (8(+2) Points)

(a) Let $A \subseteq \mathbb{R}^d$ be a closed set. Recall that a *nearest point* $x \in A$ to a given point $y \in \mathbb{R}^d$ is a point that minimizes the distance between y and the points of A with respect to the ℓ_2-norm. In class it was shown that if A is additionally convex, then nearest points are unique. Show the converse: If for every $y \in \mathbb{R}^d \setminus A$ there is a unique nearest point $x =: \pi_A(y) \in A$, then A is convex.

(b) Let C be a convex and compact set and let H be a supporting hyperplane for C at a point p in the boundary ∂C. Let $F := H \cap C$ be a "face". Show that the set of extreme points $\text{ext}F$ is contained in the set of extreme points $\text{ext}C$. This was a loophole in the proof of Minkowski's Theorem from class.

(c) Let C be compact and convex. Show that C has an extreme point. *Note:* Show this from scratch, don't simply refer to the proof of Minkowski's Theorem.

(d) *Bonus:* Prove that the set of extreme points of a closed convex set $A \subseteq \mathbb{R}^2$ is closed.

(e) Construct an example of a convex and compact set C where the set $\text{ext}C$ is not closed.

Problem 2: *Lemma from class* (6 Points)

Recall that the *support function* of a convex body $C \subset \mathbb{R}^d$ was defined as

$$\begin{aligned} h_C \colon \mathbb{R}^d &\longrightarrow \mathbb{R} \\ a &\longmapsto \max\{a^t x \colon x \in C\}. \end{aligned}$$

Prove Lemma 2.31 from class:

Let K, L and M be convex bodies. Show that

(a) $h_{K+L} = h_K + h_L$,

(b) $K + M = L + M$ implies that $K = L$.

Problem 3: *Polyhedra are spectrahedra* (6 Points)

Show that any polyhedron P, that is, the interesection of finitely many closed half-spaces in some $\mathbb{R}^d$, is a spectrahedron.