Berlin Mathematical School

Freie Universität Berlin

Prof. Günter M. Ziegler
Albert Haase, Miriam Schlöter

Institut für Mathematik
Arbeitsgruppe Diskrete Geometrie

Discrete Geometry II – Problem Sheet 6

Solution to Problem 1c

Problem 1: *Convex bodies, polars, and ellipsoids* (8(+2) Points)

(c) Let E be an ellipsoid centered at the origin given by $E = \{x \in \mathbb{R}^d \colon \langle Qx, x\rangle \leq 1\}$, where Q is a positive definite matrix. Show that $E^* = \{y \in \mathbb{R}^d \colon \langle Q^{-1}y, y\rangle \leq 1\}$.

Proof. To begin with, we will prove two facts:

- Fact 1: Given a nonempty set $K \subseteq \mathbb{R}^d$ and an invertible linear transformation A on $\mathbb{R}^d$, we have
$$(AK)^* = (A^t)^{-1}K^*,$$
where the asterisk denotes the polar.
- Fact 2: If B_d denotes the d-dimensional unit ball, then
$$B_d^* = B_d.$$

Proof of Fact 1. This was shown in class on Thursday, June 5. Let us recall it:

$$\begin{aligned}
(AK)^* &= \{y \in \mathbb{R}^d \ : \ y^tAx \leq 1 \ \forall x \in K\} \\
&= \{y \in \mathbb{R}^d \ : \ (A^ty)^tx \leq 1 \ \forall x \in K\} \\
&= \{(A^t)^{-1}A^ty \ : \ y \in \mathbb{R}^d, \ (A^ty)^tx \leq 1 \ \forall x \in K\} \\
&= \{(A^t)^{-1}z \ : \ z \in \mathbb{R}^d, \ z^tx \leq 1 \ \forall x \in K\} \\
&= (A^t)^{-1}\{z \in \mathbb{R}^d, \ : \ z^tx \leq 1 \ \forall x \in K\} \\
&= (A^t)^{-1}K^*.
\end{aligned}$$

□

Proof of Fact 2.

$$\begin{aligned}
B_d^* &= \{y \in \mathbb{R}^d \,:\, y^t x \leq 1 \; \forall x \in B_d\} \\
&= \{y \in \mathbb{R}^d \,:\, \max_{\|x\| \leq 1} y^t x \leq 1\} \\
&\subseteq \{y \in \mathbb{R}^d \,:\, \max_{\|x\| \leq 1} \|y\| \|x\| \leq 1\} \\
&= \{y \in \mathbb{R}^d \,:\, \|y\| \leq 1\} \\
&= B_d,
\end{aligned}$$

where the (third) inclusion is due to the Cauchy–Schwarz inequality. For the reverse inclusion, let $x_0 \in B_d$ and let $x \in \mathbb{R}^d$ such that $\|x\| \leq 1$, then $x_0^t x \leq \|x_0\| \|x\| \leq 1$ and hence, $x \in B_d^*$. □

The matrix Q can be written as $Q = U^t D U$ for an orthogonal matrix U and a diagonal matrix D with (strictly) positive entries. Hence Q can be written as $Q = L^t L$, where $L = \sqrt{D} U$.

Finally, let's prove the original claim in 1(c). Let

$$S := \{y \in \mathbb{R}^d \colon \langle Q^{-1} y, y \rangle \leq 1\}.$$

Let $A := L^{-1}$ and $K := B_d$ in Fact 1. Then,

$$\begin{aligned}
E^* &= (L^{-1} B_d)^* \\
&= L^t B_d^* \\
&= L^t B_d \\
&= S,
\end{aligned}$$

where the first equality is a simple calculation, the second is due to Fact 1, the third is due to Fact 2, and the last is again a simple calculation. For the simple calculations use that the inverse of a transpose is the transpose of its inverse, and the fact that inverses of symmetric matrices are symmetric. □