Berlin Mathematical School

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Discrete Geometry II – Problem Sheet 6

Please hand in your solutions to Prof. Ziegler on **Tuesday, June 03, 2014** before the lecture begins.

Problem 1: *Convex bodies, polars, and ellipsoids* (8(+2) Points)

Given a set $A \subseteq \mathbb{R}^d$, the *polar* of A, denoted by A^*, is defined as

$$A^* := \{y \in \mathbb{R}^d \colon \langle y, x\rangle \leq 1 \text{ for all } x \in A\}.$$

Let K be a (full-dimensional) convex body containing the origin in its interior.

(a) Show that K^* is a convex body containing the origin in its interior.

(b) Show that $(K^*)^* = K$.

(c) Let E be an ellipsoid centered at the origin given by $E = \{x \in \mathbb{R}^d \colon \langle Qx, x\rangle \leq 1\}$, where Q is a positive definite matrix. Show that $E^* = \{y \in \mathbb{R}^d \colon \langle Q^{-1}y, y\rangle \leq 1\}$.

(d) *Bonus:* Show that the image of an ellipsoid under a surjective affine map is again an ellipsoid.

Problem 2: *Löwner–John: maximal volume ellipsoid* (12(+2) Points)

(a) Let Δ_d be the standard d-simplex given by

$$\Delta_d := \{x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \colon x_i \geq 0 \text{ and } x_1 + \ldots x_{d+1} = 1\}.$$

View Δ_d as a d-dimensional convex body lying in the affine subspace given by the hyperplane $H := \{x \in \mathbb{R}^d \colon x_1 + \cdots + x_{d+1} = 1\}$. Determine the ellipsoid E_1 of maximal volume contained in Δ_d. If c is the center of E_1, show that the constant $\rho = d$ is minimal for the inclusion $\Delta_d - c \subseteq \rho(E_1 - c)$ to hold.[1]

(b) Let C_d be the d-cube given by

$$C_d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \colon |x_i| \leq 1\}.$$

Determine the ellipsoid E_2 of maximal volume contained in C_d.

[1] Given a set $X \subset \mathbb{R}^d$, the expression $X - c$ denotes the set $\{x - c \colon x \in X\}$. Given a positive constant δ, the expression δX refers to the set $\{\delta x \colon x \in X\}$.

(c) Let C_d^* be the standard d-dimensional octahedron given by

$$C_d^* := \{(x_1, \dots, x_d) \in \mathbb{R}^d \colon |x_1| + \dots + |x_d| \le 1\}.$$

Determine the ellipsoid E_3 of maximal volume contained in C_d^*. Prove that $\rho = \sqrt{d}$ is minimal for the inclusion $C_d^* \subseteq \rho E_3$ to hold.

(d) *Bonus:* Let C be a convex polytope such that its symmetry group G acts transitively on its set of vertices. Show that the center of the ellipsoid E of maximal volume contained in C coincides with the barycenter of the vertex set of C.[2] Does this imply that E is a ball?

[2] The action of a group G on a set X is *transitive* if there is only one orbit, that is, if $Gx = \{gx \colon g \in G\} = X$ for $x \in X$. The barycenter of a set of points $\{x_1, \dots, x_n\} \subset \mathbb{R}^d$ is the point $\sum_i \frac{1}{n} x_i$.