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Discrete Geometry II – Problem Sheet 7

Please hand in your solutions to Prof. Ziegler on **Tuesday, June 10, 2014** before the lecture begins.

Problem 1: *Original references* (6 Points)

In class we discussed theorems due to Ernst Sas, Alexander Macbeath, and György Elekes.

(a) The theorem due to Ernst Sas (1939) reads as follows:
Let C be a convex disk (a convex body in the plane) and let $n \geq 3$ be an integer. If $P_{(n)}$ is an n-gon of maximal area contained in C, and P_n^2 is a regular n-gon inscribed into the unit disk B^2, then

$$\frac{\mathrm{vol}(P_{(n)})}{\mathrm{vol}(C)} \geq \frac{\mathrm{vol}(P_n^2)}{\mathrm{vol}(B^2)} = \frac{n}{2\pi}\sin\frac{2\pi}{n},$$

with equality if and only if C is an ellipse.

(b) The theorem due to Alexander Macbeath (1951) is an extension of this result:
Let C be a convex body in $\mathbb{R}^d$ and let $n \geq d+1$ be an integer. If $P_{(n)}$ is a polytope with n vertices of maximal volume contained in C, and P_n^d is a convex polytope of maximal volume inscribed into the unit ball B^d, then

$$\frac{\mathrm{vol}(P_{(n)})}{\mathrm{vol}(C)} \geq \frac{\mathrm{vol}(P_n^d)}{\mathrm{vol}(B^d)},$$

with equality if and only if C is an ellipsoid.

(c) The theorem due to György Elekes (1986) reads as follows:
Let $P_{(n)}^d$ be a convex d-polytope with n vertices contained in B^d. Then

$$\frac{\mathrm{vol}(P_{(n)}^d)}{\mathrm{vol}(B^d)} \leq \frac{n}{2^d}.$$

For each of the theorems above, find and properly cite the original reference and say a few words about the (original) proof.

Problem 2: *Rectangular boxes* (8 Points)

(a) For every convex body C in $\mathbb{R}^d$ there is a rectangular d-box[1] R with center c and a constant $f(d) > 0$ that depends only on d such that the inclusions

$$R - c \quad \subseteq \quad C - c \quad \subseteq \quad f(d)(R - c)$$

hold.

(b) Give good upper and lower bounds for $f(d)$.

(c) What happens if we ask for axis-parallel rectangular boxes?

Problem 3: *Guarantees* (6 Points)

Show that for convex bodies $C \subseteq B^d$ the following types of "guarantees" are equivalent:

(a) C contains a ball of radius r_0, for a specified $r_0 > 0$.

(b) C has width[2] at least w_0, for a specified $w_0 > 0$.

(c) C has volume[3] at least v_0, for a specified $v_0 > 0$.

That is, show that from any such "guarantee" you can derive a guarantee for any of the two other types.

[1] A rectangular d-box is the d-fold Cartesian product of intervals.

[2] The *width* of C is defined as the minimal distance between two parallel supporting hyperplanes for C.

[3] By *volume* we mean the Lebesgue measure.