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Discrete Geometry II – Problem Sheet 8

Please hand in your solutions to Prof. Ziegler on **Tuesday, June 18, 2014** before the lecture begins.

Problem 1: *Volume of the free sum* (8 Points)

We will verify Proposition 2.61 part (3) from class in a special case. Let K and L both be simplicial convex polytopes of dimensions k respectively ℓ with the origin in their interiors. Prove that

$$\mathrm{vol}_{k+\ell}(K \oplus L) \;=\; \binom{k+\ell}{k}^{-1} \mathrm{vol}_k(K)\,\mathrm{vol}_\ell(L).$$

Problem 2: *Volume of cross sections* (6 Points)

Let $C_3 = [0,1]^3$ be the 3-dimensional unit cube. For $t \in \mathbb{R}$, define the hyperplane

$$H_t := \{x \in \mathbb{R}^3 \colon x_1 + x_2 + x_3 = t\}.$$

Now define a function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(t) := \mathrm{vol}_2(C_3 \cap H_t).$$

Explicitly express the function f in terms of t and sketch the graphs of $t \longmapsto f(t)$ and of $t \longmapsto \sqrt{f(t)}$.

Problem 3: *Signed volume* (6 Points)

For a triangle $\Delta = \Delta(a,b,c) = \mathrm{conv}\{a,b,c\}$ with ordered vertices $a,b,c \in \mathbb{R}^2$ we define the *signed volume* of Δ as

$$\mathrm{vol}_2^\circ(\Delta) \;:=\; \frac{1}{2}\det\begin{pmatrix}1 & 1 & 1\\ a & b & c\end{pmatrix}.$$

Let $P = \mathrm{conv}\{v_1,\dots,v_n\}$ be a polygon with vertices $v_i \in \mathbb{R}^2$ ordered counterclockwise. Let $p \in \mathrm{int}(P)$ be a point in the interior of P.

(a) Show the following equality where $\mathrm{vol}_2(P)$ denotes the standard 2-dimensional volume of P:

$$\mathrm{vol}_2(P) = \sum_{i=1}^{n} \mathrm{vol}_2^{\circ}(\Delta(p, v_i, v_{i+1})) \quad \text{where } v_{n+1} := v_1. \tag{$*$}$$

(b) Argue geometrically that the right-hand side of $(*)$ is independent of the choice of $p \in \mathbb{R}^2$ — whether $p \in P$ or not.
(This shows that the volume is a polynomial of the vertex coordinates and invariant under translation).