Berlin Mathematical School
Freie Universität Berlin

Prof. Günter M. Ziegler
Albert Haase, Miriam Schlöter

Institut für Mathematik
Arbeitsgruppe Diskrete Geometrie

Discrete Geometry II – Problem Sheet 8

Please hand in your solutions to Prof. Ziegler on **Tuesday, June 18, 2014** before the lecture begins.

Problem 1: *Equality* (6 Points)

Let K and L be d-dimensional polyboxes in $\mathbb{R}^d$, that is, nonempty finite unions of d-dimensional axis-parallel rectangular boxes. Let's say K and L are positively homothetic, that is, there is a $\lambda > 0$ such that $L = \lambda K + c$ for some $c \in \mathbb{R}^d$. Is the Brunn-Minkowski Inequality

$$\sqrt[d]{\mathrm{vol}_d(K+L)} \geq \sqrt[d]{\mathrm{vol}_d(K)} + \sqrt[d]{\mathrm{vol}_d(L)}$$

satisfied with equality?

Problem 2: *Multiplicative Brunn–Minkowski* (8 Points)

(a) Prove the *weighted* arithmetic-geometric-mean inequality

$$\sum_{i=1}^{n} \lambda_i z_i \geq \prod_{i=1}^{n} z_i^{\lambda_i}$$

for all $z_i \geq 0$ and $\lambda_i \in [0,1]$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

(b) Prove the *multiplicative* Brunn-Minkowski inequality

$$\mathrm{vol}_d((1-\lambda)K + \lambda L) \geq \mathrm{vol}_d(K)^{1-\lambda}\, \mathrm{vol}_d(L)^{\lambda}$$

for all convex bodies $K, L \subset \mathbb{R}^d$ and $\lambda \in [0,1]$.

(c) What are the equality cases?

Problem 3: *Cayley Embedding* (6 Points)

Let $K_1, \ldots, K_n \subset \mathbb{R}^d$ be convex bodies. The *Cayley embedding* of $(K_i : i = 1, \ldots, n)$ is defined as

$$C := \mathrm{Cay}(K_1, K_2, \ldots, K_n) := \mathrm{conv}\{(p_i, e_i) \ : \ p_i \in K_i, \ i = 1, \ldots, n\} \subset \mathbb{R}^d \times \mathbb{R}^n,$$

where e_i denotes the i-th unit vector in $\mathbb{R}^n$. (The Cayley embedding for two sets was used in the proof of Brunn's Slice Inequality.)

(a) Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ with $\lambda_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_i \lambda_i = 1$. Show that
$$C \cap \{(x, y) \ : \ y = \lambda\} \cong \sum_{i=1}^{n} \lambda_i K_i.$$

(b) Show that for every proper subset $I \subset [n]$ the Cayley embedding $\mathrm{Cay}(K_i : i \in I)$ is a face of C.

(c) For $m \geq n$, let $P \subset \mathbb{R}^m$ be an m-polytope and let $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a linear projection. Show that if $Q := \pi(P)$ is an $(n-1)$-simplex and every vertex of P is mapped to a vertex of Q, then P is linearly isomorphic to $\mathrm{Cay}(P_1, \ldots, P_n)$ for some polytopes $P_1, \ldots, P_n$.