Berlin Mathematical School

Freie Universität Berlin

Prof. Günter M. Ziegler
Albert Haase, Miriam Schlöter

Institut für Mathematik
Arbeitsgruppe Diskrete Geometrie

Discrete Geometry II – Problem Sheet 10

Please hand in your solutions to Prof. Ziegler on **Tuesday, July 1, 2014** before the lecture begins.

Problem 1: *Facet Volume and Polytope Volume* (8 Points)

In the lecture we proved the equality

$$v(b) := \mathrm{vol}_d(P_A(b)) = \frac{1}{d}\sum_{i=1}^{n} \mathrm{vol}_{d-1}\left(F_i(P_A(b))\right) \cdot b_i.$$

Here $A \in \mathbb{R}^{n\times d}$ for $n > d$ is a matrix with full rank and pairwise distinct row vectors that all have length one. The polytope

$$P_A(b) = \{x \in \mathbb{R}^d : Ax \leq b\},$$

for $b \in \mathbb{R}^n$, has facets $F_i(P_A(b))$ with normal vector equal to the i-th column of A. The set of b-vectors for which $P_A(b)$ is non-empty is

$$\mathcal{B}_A := \{b \in \mathbb{R}^n : P_A(b) \neq \emptyset\};$$

the set of b-vectors that yield polytopes of volume at least 1 is

$$\mathcal{M}_A := \{b \in B_A : \mathrm{vol}(P_A(b)) \geq 1\}.$$

As an example, define the matrix

$$A := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}.$$

Choose a coordinate system (= basis) for $\mathbb{R}^5/\operatorname{im}(A) \cong \mathbb{R}^2$.

(a) Sketch $\overline{\mathcal{B}_A} := \mathcal{B}_A/\operatorname{im}(A)$ and $\overline{\mathcal{M}_A} := \mathcal{M}_A/\operatorname{im}(A)$.

(b) Explicitly determine $\overline{v} : b \overset{\overline{v}}{\longmapsto} v(b)$ as a piecewise polynomial function $\overline{\mathcal{B}_A} \to \mathbb{R}$.

(c) Plot the function $\overline{v}$.

(d) Is $\bar{v}$ differentiable on the interior of its domain $\mathcal{B}_A$? Twice differentiable?

Problem 2: *Volume of Minkowski Sum of Polytope and Scaled Ball* (6 Points)

Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope and let $B_d \subset \mathbb{R}^d$ be the unit ball. Define the function

$$f_P(t) := \mathrm{vol}_d(P + tB_d).$$

(a) For $d = 2$ show that $f_P(t)$ is a polynomial of degree 2 and give an interpretation of its coefficients.

(b) Let $d = 2$. Show that the Brunn–Minkowski inequality for P and B_2 is equivalent to the "isoperimetric inequality" $p^2 \geq 4\pi a$, where p is the perimeter and a is the area of P.

(c) Let $d = 3$. Show that $f_P(t)$ is a polynomial of degree 3 and interpret its coefficients.

Problem 3: *Orthant-Shaped Polyhedra* (6(+2) Points)

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^d_{>0}$ have strictly positive coordinates and let $b \in \mathbb{R}^n_{\geq 0}$ have non-negative coordinates. Define the *orthant-shaped polyhedron*

$$Q(b) := \{x \in \mathbb{R}^d : x \geq 0,\ a_i^t x \geq b_i \text{ for all } i = 1, \dots, n\}.$$

For $\alpha \in \mathbb{R}^n_{>0}$, define

$$M := \{b \in \mathbb{R}^n_{\geq 0} : \mathrm{vol}_{d-1} F_i(Q(b)) \leq \alpha_i \text{ for all } i = 1, \dots, n\},$$

where $F_i(Q(b))$ denotes the facet of $Q(b)$ with normal vector a_i.

(a) Show that for $i = 1, \dots, n$ and $\varepsilon > 0$

$$\mathrm{vol}_{d-1}\big(F_i(Q(b))\big) \leq \mathrm{vol}_{d-1}\big(F_i(Q(b + \varepsilon e_i))\big),$$

where e_i denotes the i-th unit vector in $\mathbb{R}^n$.

(b) Show that there is a constant $c > 0$ such that $b_i \leq c$ for all $b \in M$.

(c) Deduce from (b) that there is a constant $u > 0$ such that

$$P(b) := Q(b) \cap \{x \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d \leq u\}$$

is a polytope such that $F_i(P(b)) = F_i(Q(b))$.

(d) *Bonus:* Show that M is convex.