Berlin Mathematical School

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Discrete Geometry II – Problem Sheet 11

Please hand in your solutions to Prof. Ziegler on **Tuesday, July 8, 2014** before the lecture begins.

Problem 1: *An Order Polytope* (8 Points)

Let $n \geq 2$, and let $(X, \preceq)$ be a disjoint union of an $(n-1)$-chain and an element. For example, let $X = \{a_1, a_2, \ldots, a_n$ with $a_i \prec a_j$ iff $1 < i < j\}$. The *order polytope* of $(X, \preceq)$ is

$$P(X, \preceq) = \{x \in [0,1]^n : x_i \leq x_j \text{ for } a_i \preceq a_j\}.$$

(a) Describe $P(X, \preceq)$: What is its dimension, its number of facets, its number of vertices, its volume?

(b) How many pairwise comparisons do you need for sorting the poset $(X, \preceq)$? What upper bound for the number of steps do you get from the approach/proof discussed in class?

Problem 2: *Type Cones and Minkowski Addition* (6(+2) Points)

Let $A \in \mathbb{R}^{n \times d}$ be a rank d matrix for which the sum of the rows is zero.

(a) Show that for any $b \in \mathbb{R}^n$, the polyhedron $P_A(b)$ is bounded.
(More generally, this holds if and only if there are positive values α_i such that $\alpha_1 a_1 + \ldots + \alpha_n a_n = 0$.)

(b) Give an example where $P_A(b') + P_A(b'') \neq P_A(b' + b'')$.
If there is no equality, which inclusion fails?

(c) *Bonus:* Show that $P_A(b') + P_A(b'') = P_A(b' + b'')$ if and only if b' and b'' lie in the closure of the same type cone (namely, the type cone $\mathcal{T}_A(b' + b'')$).

Problem 3: *An Inner Type Cone* (8 Points)

(a) For the matrix

$$A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

describe

(i) the cone $\mathcal{B}_A$ of all right-hand side vectors $b \in \mathbb{R}^5$ such that $P_A(b) \neq \emptyset$, Is this a polyhedral cone? What is its dimension? How many facets/rays does it have?

(ii) the closed inner region $\mathcal{B}_A^\circ$ of all right-hand side vectors $b \in \mathbb{R}^5$ such that all five inequalities define non-empty faces of $P_A(b)$,
What is its dimension? Is this a polyhedral cone? If yes, how many facets/rays does it have?

(iii) the other type cones that $\mathcal{B}_A$ decomposes into? Sketch the decomposition.

(b) Argue that for a general matrix A, the inner type cone is a polyhedral cone. Give an upper bound for its number of facets.
(Hint: When does an inequality become redundant? If it does, due to "too large right-hand side", then some version of the Farkas lemma will give you a certificate, which you can use.)