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Discrete Geometry II – Problem Sheet 12

Please hand in your solutions to Prof. Ziegler on **Tuesday, July 15, 2014** before the lecture begins. **This the last sheet. It has 6 bonus points.**

Problem 1: *Mixed Volume of two Ellipses* (6(+6) Points)

Let K and L be two ellipses in $\mathbb{R}^2$, say,

$$K := \{(x,y) \in \mathbb{R}^2 : \tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} \le 1\},$$
$$L := \{(x,y) \in \mathbb{R}^2 : \tfrac{x^2}{c^2} + \tfrac{y^2}{d^2} \le 1\}.$$

(a) Is the Minkowski sum $K + L$ an ellipse?

(b) Determine the function

$$f(\lambda_1, \lambda_2) = \mathrm{vol}_2(\lambda_1 K + \lambda_2 L).$$

How many values of f do you need in order to determine f?

(c) *Bonus:* Compute the mixed volume

$$\mathrm{MV}(K, L).$$

Problem 2: *Mixed Volumes and a One-Dimensional Projection* (8 Points)

Let $u \in \mathbb{R}^d \setminus \{0\}$ and let $[0,u] := \{\lambda u : 0 \le \lambda \le 1\}$. Let $u^\perp$ denote the linear hyperplane perpendicular to u.

(a) Show that for a convex body $K \subset \mathbb{R}^d$

$$\mathrm{MV}\big(K[d-1], [0,u]\big) = \frac{\|u\|}{d}\,\mathrm{vol}_{d-1}\big(\pi_u(K)\big).$$

Here $\pi_u \colon \mathbb{R}^d \to u^\perp$ is the orthogonal projection onto $u^\perp$.

(b) Show that the function $f \colon \mathbb{R}^d \to \mathbb{R}$ given by

$$f(u) := \mathrm{MV}\big(K[d-1], [0,u]\big)$$

is convex. [*Hint:* First observe that f is positively linear and then show that $f(u+v) \le f(u) + f(v)$.]

Problem 3: *Volume of a Zonotope* (8 Points)

For vectors $z_1, z_2, \dots, z_n \in \mathbb{R}^d \setminus \{0\}$ and using the notation of Problem 2 above define the *zonotope*

$$Z := [0, z_1] + [0, z_2] + \cdots + [0, z_n].$$

(a) Show that

$$\mathrm{vol}_d(Z) \quad = \sum_{1 \le i_1 < i_2 < \cdots < i_d \le n} |\det(z_{i_1}, z_{i_2}, \dots, z_{i_d})| \, .$$

(b) For π_u as in Problem 2, consider the function

$$f(u) := \mathrm{vol}_{d-1}(\pi_u(Z)).$$

Show that there is a zonotope $\Pi_Z = \sum_{j=1}^{M} [-w_j, w_j]$ such that

$$f(u) = \max\{u^t x : x \in \Pi_Z\} \quad \text{for all } u \in \mathbb{S}^{d-1}.$$