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Topologie II – Exercise Sheet 1

Date of assignment: **Wednesday, Oct. 15, 2014**. We highly recommend problems marked with a star. Do the other exercises if they seem challenging enough or if you don't have an idea of how to solve them immediately.

Exercise 1: *Subgroups of Free Abelian Groups*

Show that any subgroup of a free abelian group is itself a free abelian group. Also, recall the fundamental difference between "free group" and "free abelian group" and choose a set S such that the free group generated by S and the free abelian group generated by S are non-isomorphic.

Exercise 2: *Cyclic Groups and Fundamental Theorem*

Recall the Fundamental Theorem of Finitely Generated Abelian Groups as stated in the tutorial:

Given a finitely generated abelian group G, there exists a decomposition $G \cong H \oplus T$, where T denotes the torsion subgroup of G and H denotes a free abelian group of finite rank β. Furthermore, $T \cong \mathbb{Z}/t_1 \oplus \mathbb{Z}/t_2 \oplus \cdots \oplus \mathbb{Z}/t_k$ for $t_i \in \mathbb{Z}_{>1}$ such that $t_1|t_2|\ldots|t_k$ (successively divide). The t_i and β are uniquely determined by G.[1]

(a) Show that $\mathbb{Z}/m \oplus \mathbb{Z}/n \cong \mathbb{Z}/mn$ if and only if m and n are coprime.

(b) For the following finitely generated abelian groups, calculate their decomposition according to the above theorem.

   (i) $\mathbb{Z}/2 \oplus \mathbb{Z}/5 \oplus \mathbb{Z} \oplus \mathbb{Z}/2$.

   (ii) $\mathbb{Z}/5 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/9$.

   (iii) $\mathbb{Z}/5 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/2$.

[1] See J. Munkres, *Elements of Algebraic Topology*, Ch. 1 Sec. 4 Th. 4.3. This section also explains the algebraic backround.

Exercise 3: *Definition of Categories*

(a) Let $\mathcal{C}$ be a category and let $A \in \operatorname{obj} \mathcal{C}$. Prove that $\hom(A, A)$ has a unique identity 1_A.

(b) If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and if $A \in \operatorname{obj} \mathcal{C}'$, then the identity in $\hom_{\mathcal{C}'}(A, A)$ is equal to the identity in $\hom_{\mathcal{C}}(A, A)$.

*Exercise 4: *Examples of Categories*

(a) Let G be a monoid (with a neutral element). Show that the following construction gives a category $\mathcal{C}$. Let $\operatorname{obj} \mathcal{C} = \{*\}$, hence consist of one element. Define $\hom(*, *) = G$ and define the composition by group multiplication. This example shows that morphisms need not be functions.

(b) Given a category $\mathcal{C}$, show that the following construction gives a category $\mathcal{M}$, called a *morphism category*. The objects of $\mathcal{M}$ are the morphisms of $\mathcal{C}$. Next, if $f, g \in \operatorname{obj} \mathcal{M}$ such that $f \in \hom(A, B)$ and $g \in \hom(C, D)$, then a morphism in $\hom(f, g)$ is a pair (h, k) of morphisms in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

(c) is well-defined and commutes. Define the composition coordinate-wise, that is, $(h', k') \circ (h, k) = (h' \circ h, k' \circ k)$.

(d) Let G be a group and let $\mathcal{C}$ be the category associated to it in part (a). If H is a normal subgroup of G, define a relation by $x \sim y$ if and only if $xy^{-1} \in H$. Show that $\sim$ leads to an equivalence on the category $\mathcal{C}$ and that for the corresponding quotient category $\mathcal{C}'$ we have $[*, *] = G/H$.

***Exercise 5:** *Examples of Functors*

(a) Given a category $\mathcal{C}$, prove that for a fixed object $M \in \operatorname{obj}\mathcal{C}$, the mapping that sends $A \in \operatorname{obj}\mathcal{C}$ to $\operatorname{Hom}(M,A) = \hom(M,A)$ respectively $\operatorname{Hom}(A,M) = \hom(A,M)$ is a covariant respectively contravariant functor from $\mathcal{C}$ to the category **Sets**. To prove this, first define $f \longmapsto \operatorname{Hom}(M,f)$ and $f \longmapsto \operatorname{Hom}(f,M)$ for $f \in \hom_{\mathcal{C}}(A,B)$ and $A, B \in \operatorname{obj}\mathcal{C}$ in a suitable way.

(b) In the above setting for $\mathcal{C} = \mathbf{Groups}$ and $C \in \mathcal{C}$ and $g \in \hom_{\mathcal{C}}(B,C)$, let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

be an exact sequence[2] of groups. In the following we assume that both Hom functors are functors from **Groups** to **Groups**. In order to speak of exact sequences we need the target category to be a so-called *abelian category*. Show that

(i) $0 \longrightarrow \operatorname{Hom}(M,A) \xrightarrow{\operatorname{Hom}(M,f)} \operatorname{Hom}(M,B) \xrightarrow{\operatorname{Hom}(M,g)} \operatorname{Hom}(M,C)$

is exact.

(ii) $\operatorname{Hom}(A,M) \xleftarrow[\operatorname{Hom}(f,M)]{} \operatorname{Hom}(B,M) \xleftarrow[\operatorname{Hom}(g,M)]{} \operatorname{Hom}(C,M) \longleftarrow 0$

is exact.

Note that the above shows that both Hom-functors are *left-exact*.

(c) For an abelian group G let T_G be its torsion subgroup.

(i) Show that $G \overset{t}{\longmapsto} T_G$ defines a functor from $\mathbf{Ab} \longrightarrow \mathbf{Ab}$ if we define $t(f) := f|T_G$ (restriction) for every $f \in \hom(G,H)$ for $G, H \in \mathbf{Ab}$.

(ii) Show that if f is injective, then $t(f)$ is injective. Phrase this in terms of "exactness of funtors".

(iii) Show that f surjective does not imply $t(f)$ surjective. Phrase this in terms of "exactness of funtors".

[2] If "kernel" and "image" are well-defined in a category, then an *exact sequence* in that category is a sequence of objects and morphisms such that for each morphism its image is equal to the kernel of the next morphism.