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Topologie II – Exercise Sheet 1

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*Exercise 4: *Examples of Categories*

(a) Let G be a monoid (with a neutral element). Show that the following construction gives a category $\mathcal{C}$. Let $\operatorname{obj}\mathcal{C} = \{*\}$, hence consist of one element. Define $\hom(*, *) = G$ and define the composition by group multiplication. This example shows that morphisms need not be functions.

Solution:

- We have a class of homomorphisms for every object (there is only the object $*$) and a composition law defined by the group-operation.
- The families of homomorphisms are clearly pairwise disjoint, since there is only one such family.
- The composition law is associative, since the group-operation is associative.
- The identity morphism $\mathbb{1}_*$ in $\hom(*, *)$ is given by the neutral element $e \in G$ since it satisfies $eg = ge$ for all $g \in G = \hom(*, *)$.

(b) Given a category $\mathcal{C}$, show that the following construction gives a category $\mathcal{M}$, called a *morphism category*. The objects of $\mathcal{M}$ are the morphisms of $\mathcal{C}$. Next, if $f, g \in \operatorname{obj}\mathcal{M}$ such that $f \in \hom(A, B)$ and $g \in \hom(C, D)$, then a morphism in $\hom(f, g)$ is a pair (h, k) of morphisms in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

is well-defined and commutes. Define the composition coordinate-wise, that is, $(h', k') \circ (h, k) = (h' \circ h, k' \circ k)$.

Solution:

- We have a set of morphisms for every object and a composition law for any two morphisms (as defined on the sheet).
- We regard a morphism (h, k) together with its "source", and "target". In other words: if $(h, k) \in \hom_{\mathcal{M}}(f, g)$, then as in the case of the category $\mathcal{C}$, (h, k) is not only given by the objects A, B, C, D and $\mathcal{C}$-morphisms h, k, but also by the source object f and the target object g. The fact that the morphism-classes in $\mathcal{M}$ are disjoint follows immediately from this fact.
- The composition law in $\mathcal{M}$ is associative, since the composition law in $\mathcal{C}$ is associative.
- Given an object $f \in \text{obj}(\mathcal{M})$ with $f \in \hom_{\mathcal{C}}(A, B)$ the identity in $\hom_{\mathcal{M}}(f, f)$ is given by $\text{id}_f := (\text{id}_{A,A}, \text{id}_{B,B})$, where $\text{id}_{A,A} \in \hom_{\mathcal{C}}(A, A)$ is the identity.
- Given $f \in \text{obj}(\mathcal{M})$ such that $f \in \hom_{\mathcal{C}}(A, B)$, then $\text{id}_f \circ (h, k) = (\text{id}_A \circ h, \text{id}_B \circ k) = (h, k)$ for all $(h, k) \in \hom(e, f)$ and all $e \in \text{obj}(\mathcal{M})$. And $(h', k') \circ \text{id}_f = (h' \circ \text{id}_A, k' \circ \text{id}_B) = (h', k')$ for every $(h', k') \in \hom(f, g)$ and all $g \in \text{obj}(\mathcal{M})$.

(c) Let G be a group and let $\mathcal{C}$ be the category associated to it in part (a). If H is a normal subgroup of G, define a relation by $x \sim y$ if and only if $xy^{-1} \in H$. Show that $\sim$ leads to an equivalence on the category $\mathcal{C}$ and that for the corresponding quotient category $\mathcal{C}'$ we have $[*, *] = G/H$.

Solution:

- Let $f \in \hom_{\mathcal{C}}(*, *)$ and $f \sim f'$, then $f' \in \hom_{\mathcal{C}}(*, *)$, since there is only one set of morphisms.
- Let $f \sim f'$ and $g \sim g'$ and let gf exist. Then $(gf)(g'f')^{-1} = gff'^{-1}g'^{-1} \in H$ since $ff'^{-1} \in H$ and H is a normal subgroup of G.
- Next we will show that the set of morphisms $[*, *]$ in $\mathcal{C}'$, the quotient category, is equal to G/H. By definition $[*, *] = \{[f] \ : \ f \in \hom_{\mathcal{C}}(*, *)\}$. The set on the right hand side is precisely the set of all cosets of H in G and hence $[*, *] = G/H$.

***Exercise 5:** *Examples of Functors*

(a) Given a category $\mathcal{C}$, prove that for a fixed object $M \in \operatorname{obj}\mathcal{C}$, the mapping that sends $A \in \operatorname{obj}\mathcal{C}$ to $\operatorname{Hom}(M, A) = \operatorname{hom}(M, A)$ respectively $\operatorname{Hom}(A, M) = \operatorname{hom}(A, M)$ is a covariant respectively contravariant functor from $\mathcal{C}$ to the category **Sets**. To prove this, first define $f \longmapsto \operatorname{Hom}(M, f)$ and $f \longmapsto \operatorname{Hom}(f, M)$ for $f \in \operatorname{hom}_{\mathcal{C}}(A, B)$ and $A, B \in \operatorname{obj}\mathcal{C}$ in a suitable way.

Solution of (a):

Let (C) be a category and let $M \in \operatorname{obj}(\mathcal{C})$ be a fixed object.

Part 1: Show that $A \longmapsto \operatorname{hom}(M, A)$ for $A \in \operatorname{obj}(\mathcal{C})$ defines a covariant functor from $\mathcal{C}$ to **Sets**.

(i) If $A \in \operatorname{obj}(\mathcal{C})$, then $\operatorname{Hom}(M, A)$ is a set by definition of the category $\mathcal{C}$.

(ii) Given $f \in \operatorname{hom}_{\mathcal{C}}(A, A')$ for $A, A' \in \operatorname{obj}(\mathcal{C})$, define
$\operatorname{Hom}(M, f) \in \operatorname{hom}_{\mathbf{Sets}}(\operatorname{Hom}_{\mathcal{C}}(M, A), \operatorname{Hom}_{\mathcal{C}}(M, A'))$ by
$\operatorname{Hom}(M, f)(g) := f \circ g$ for $g \in \operatorname{hom}_{\mathcal{C}}(M, A)$.

(iii) Let $f \in \operatorname{hom}_{\mathcal{C}}(A, B)$ and $f' \in \operatorname{hom}_{\mathcal{C}}(B, C)$ for $A, B, C \in \operatorname{obj}(\mathcal{C})$. Then:
$\operatorname{Hom}(M, f \circ f')(g) = (f' \circ f) \circ g = f' \circ (f \circ g) = \operatorname{Hom}(M, f') \circ \operatorname{Hom}(M, f)(g)$
for $g \in \operatorname{hom}_{\mathcal{C}}(A, B)$

(iv) Given $A \in \operatorname{obj}(\mathcal{C})$, then $\operatorname{Hom}(M, 1_A)(g) = 1_A \circ g = g$ for all $g \in \operatorname{hom}_{\mathcal{C}}(A, A)$.

Part 2: Show that $A \longmapsto \operatorname{hom}(A, M)$ for $A \in \operatorname{obj}(\mathcal{C})$ defines a contravariant functor from $\mathcal{C}$ to **Sets**.

(i) If $A \in \operatorname{obj}(\mathcal{C})$, then $\operatorname{Hom}(A, M)$ is a set by definition of the category $\mathcal{C}$.

(ii) Given $f \in \operatorname{hom}_{\mathcal{C}}(A, A')$ for $A, A' \in \operatorname{obj}(\mathcal{C})$, define
$\operatorname{Hom}(f, M) \in \operatorname{hom}_{\mathbf{Sets}}(\operatorname{Hom}_{\mathcal{C}}(A', M), \operatorname{Hom}_{\mathcal{C}}(A, M))$ by
$\operatorname{Hom}(f, M)(g) := g \circ f$ for $g \in \operatorname{hom}_{\mathcal{C}}(A', M)$.

(iii) as in Part 1 (iii) "with arrows reversed".

(iv) as in Part 1 (iv) "with arrows reversed".

(b) In the above setting for $\mathcal{C} = \mathbf{Groups}$ and $C \in \mathcal{C}$ and $g \in \operatorname{hom}_{\mathcal{C}}(B, C)$, let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

be an exact sequence[1] of groups. In the following we assume that both Hom functors are functors from **Groups** to **Groups**. In order to speak of exact sequences we need the target category to be a so-called *abelian category*. Show

[1] If "kernel" and "image" are well-defined in a category, then an *exact sequence* in that category is a sequence of objects and morphisms such that for each morphism its image is equal to the kernel of the next morphism.

that

(i) $$0 \longrightarrow \mathrm{Hom}(M,A) \xrightarrow{\mathrm{Hom}(M,f)} \mathrm{Hom}(M,B) \xrightarrow{\mathrm{Hom}(M,g)} \mathrm{Hom}(M,C)$$
is exact.

(ii) $$\mathrm{Hom}(A,M) \xleftarrow[\mathrm{Hom}(f,M)]{} \mathrm{Hom}(B,M) \xleftarrow[\mathrm{Hom}(g,M)]{} \mathrm{Hom}(C,M) \longleftarrow 0$$
is exact.

Note that the above shows that both Hom-functors are *left-exact*.

Solution of (b):

Proof of (i):

(1) We first show that $\ker(\mathrm{Hom}(M,f))$ is trivial. Let $h \in \mathrm{Hom}(M,A)$ such that $\mathrm{Hom}(M,f)(h) = f \circ h = 0$. Assume that $h \neq 0$, then there is some $x \in M$ such that $h(x) \neq 0$. Hence $f(g(x)) = 0$ is contradicting that $\ker(f) = 0$.

(2) We now show that $\mathrm{im}(\mathrm{Hom}(M,f)) = \ker(\mathrm{Hom}(M,g))$ holds.

"$\subseteq$": Let $h \in \mathrm{im}(\mathrm{Hom}(M,f))$, then $h = f \circ h'$ for some $h' \in \mathrm{Hom}(M,A)$. Hence $\mathrm{Hom}(M,g)(h) = g \circ h = g \circ f \circ h' = 0$, by the exactness of

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0. \qquad (*)$$

"$\supseteq$": Let $h' \in \ker(\mathrm{Hom}(M,g))$, then $\mathrm{Hom}(M,g)(h') = g \circ h' = 0$. Then $g(h'(x)) = 0$ for all $x \in M$. By the exactness of $(*)$ choose for every $x \in M$ a $y \in A$ such that $f(y) = h'(x)$. This defines a map $h : M \longrightarrow A$. It is a homomorphism because h' is a homomorphism. Also $\mathrm{Hom}(M,f)(h) = h'$.

Proof of (ii):

(1) We first show that $\ker(\mathrm{Hom}(g,M))$ is trivial. So let $h \in \mathrm{Hom}(C,M)$ such that $\mathrm{Hom}(g,M)(h) = h \circ g = 0$. Assume that $h \neq 0$, then there is some $x \in C$ such that $h(x) \neq 0$. Hence $h(g(x)) = 0$ is contradicting that $\ker(g) = 0$.

(2) We now show that $\mathrm{im}(\mathrm{Hom}(g,M)) = \ker(\mathrm{Hom}(f,M))$ holds.

"$\subseteq$": Let $h \in \mathrm{im}(\mathrm{Hom}(g,M))$, then $h = h' \circ g$ for some $h' \in \mathrm{Hom}(C,M)$. Hence $\mathrm{Hom}(f,M)(h) = h \circ f = h' \circ g \circ f = 0$ by the exactness of $(*)$.

"$\supseteq$": Let $h' \in \ker(\mathrm{Hom}(f,M))$, then $\mathrm{Hom}(f,M)(h') = h' \circ f = 0$. Then $h'(f(x)) = 0$ for all $x \in M$. By the exactness of $(*)$ define the map $h : C \longrightarrow M$ as $h = h' \circ g^{-1}$. This h is a well-defined homomorphism since $\mathrm{im}(g) = \ker(0) = C$ and $\mathrm{Hom}(g,M)(h) = h' \circ g^{-1} \circ g = h'$ holds.

(c) For an abelian group G let T_G be its torsion subgroup.

(i) Show that $G \stackrel{t}{\longmapsto} T_G$ defines a functor from **Ab** $\longrightarrow$ **Ab** if we define

$t(f) := f|T_G$ (restriction) for every $f \in \hom(G, H)$ for $G, H \in \mathbf{Ab}$.

(ii) Show that if f is injective, then $t(f)$ is injective. Phrase this in terms of "exactness of funtors".

(iii) Show that f surjective does not imply $t(f)$ surjective. Phrase this in terms of "exactness of funtors".

Solution of (c):

Part (i):

- Certainly T_G is an abelian group for any abelian group G.
- Let $f : G \longrightarrow G'$ be a homomorphism of groups, then $t(f) := f|T_G$. Given an element $a \in T_G$, f will map it to an element of finite order, hence $f(a) \in T_{G'}$ and $t(f)$ is well-defined.
- Let $G \xrightarrow{f} G' \xrightarrow{g} G''$ be two homomorphisms of abelian groups, then $t(g \circ f) = t(g) \circ t(f)$ by associativity of the composition of group homomorphisms.
- Let G be an abelian group and $\mathrm{id} : G \longrightarrow G$ be the identity on G, then $t(\mathrm{id}) : T_G \longrightarrow T_G$ is the identity on T_G.

Part (ii):

- Let $f : G \longrightarrow G'$ be a homomorphism of abelian groups s.t. $\ker(f) = 0$. Assume there is an $x \in T_G$ s.t. $t(f)(x) = 0$. This implies that $f(x) = 0$ and hence $x = 0$. Hence $\ker(t(f)) = 0$.
- $0 \longrightarrow G \longrightarrow G'$ exact implies that $0 \longrightarrow T_G \longrightarrow T_{G'}$ is exact.

Part (iii):

Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}/2$ be given by $f(1) = 1$. This is easily seen to be a homomorphism of abelian groups. Also it is surjective. We have $t(\mathbb{Z}) = 0$ and $t(\mathbb{Z}/2) = \mathbb{Z}/2$, hence $t(f) : 0 \longrightarrow \mathbb{Z}/2$ is the inclusion which is not surjective.