Berlin Mathematical School

Freie Universität Berlin

Prof. Pavle Blagojević — Arbeitsgruppe Diskrete Geometrie

Albert Haase

Prof. Holger Reich — Arbeitsgruppe Algebraische Topologie

Topologie II – Exercise Sheet 2

Date of assignment: **Monday, Oct. 28, 2014**.

*Exercise 2: *Short Exact Sequences and Ranks*

Let $\mathcal{C}$ be an abelian category. An exact sequence is called a *short exact sequence (SES)* if it is an exact sequence of the form

$$0 \xrightarrow{i} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\pi} 0 \qquad (*)$$

where 0 is the *zero object* and A, B, C are objects in $\mathcal{C}$ such that the morphisms as indicated have the property that the image of one morphism is equal to the kernel of the next morhpism. Since the zero object is both *terminal* and *initial*, the morphisms from 0 to A and from C to 0 are uniquely determined by A and C.

(a) Show that f is injective (define this first).

(b) Show that g is surjective (define this first).

(c) Assume only for part (c) that C is equal to the zero object. Show that f is an isomorphism (define this first). Notation: $A \cong B$ via f.

Assume for the remaining exercise that $\mathcal{C} = \mathbf{Ab}$ (the category of abelian groups).

(d) Show that $\operatorname{im} f \cong A$ and that $B/\operatorname{im} f \cong C$.

(e) Define the *rank* $\operatorname{rk} G$ of an abelian group G as the cardinality of a maximally $\mathbb{Z}$-linearly independet subset (this is well defined!). Assume that A, B, C have finite rank. Prove that $\operatorname{rk} B = \operatorname{rk} A + \operatorname{rk} C$. Draw an analogy to the dimension formulas for linear maps between vector spaces and the quotient vector space!

Solution:

(a) A morphism f in an abelian category is called injective if $\ker(f) = 0$. Since $(*)$ is exact, $0 = \operatorname{im}(i) = \ker(f)$.

(b) A morphism f in an abelian category is called surjective if $\operatorname{im}(f)$ is equal to the target object of f. Since $(*)$ is exact, $\operatorname{im}(g) = \ker(\pi) = 0$.

(c) In an abelian category a morphism is called an isomorphism if it is both injective an surjective. If

$$0 \xrightarrow{i} A \xrightarrow{f} B \xrightarrow{\pi} 0$$

is exact, then $0 = \mathrm{im}(i) = \ker(f)$ and $\mathrm{im}(f) = \ker(\pi) = B$.

(d) (1) We show that $\mathrm{im}(f) \simeq A$: Define the homomorphism $\tilde{f} : A \longrightarrow \mathrm{im}(f)$ by $a \longmapsto f(a)$. Since $0 \in \mathrm{im}(f)$ we get $\ker(f) = \ker(\tilde{f}) = 0$. Furthermore $\tilde{f}$ is trivially surjective. Hence $\mathrm{im}(f) \simeq A$ via $\tilde{f}$.

(2) We show that $B/\mathrm{im}(f) \simeq C$. We define the map $\tilde{g} : B/\mathrm{im}(f) \longrightarrow C$ with $\tilde{g}(x + \mathrm{im}(f)) = g(x)$. Then $\tilde{g}$ is a homomorphism since g is a homomorphism. Let $x \in \ker(\tilde{g})$, then $x \in \ker(g)$, hence $x \in \mathrm{im}(f)$ and hence $x + \mathrm{im}(f) = 0 + \mathrm{im}(f)$. Finally, g is surjective since $\tilde{g}$ is surjective.

(e) Suppose first that $a_1, ..., a_k$ are independent in A and $c_1, ..., c_l$ are independent in C. Let $\bar{a}_j := f(a_j)$ and choose $\bar{c}_j$ such that $g(\bar{c}_j) = c_j$. We show that $\bar{a}_1, ..., \bar{a}_k, \bar{c_1}, ..., \bar{c_l}$ are independent in B. Suppose $\sum m_i \bar{a}_i + \sum n_j \bar{(c)}_j = 0$ in B. Applying g, this implies $\sum n_j c_j = 0$, so all the $n_j' s$ are 0, but then $\sum m_i \bar{a}_i = 0$. Since f is injective $\sum m_i a_i = 0$ and so all the $m_i' s$ are 0. Therefore $\mathrm{rk}(B) \geq \mathrm{rk}(A) + \mathrm{rk}(C)$.

Suppose that $b_1, ..., b_r$ are independent in B. Let s denote the largest number of the elements $g(b_i)$ that are independent in C. After renumbering we can take $g(b_1), ..., g(b_s)$ independent in C while $g(b_1), ..., g(b_s), g(b_t)$ are dependent for any $t > s$. That is, for each $t > s$ we have a relation $\sum_{i=1,..,s} n_{it} g(b_i) = n_t g(b_t) = 0$with $n_t \neq 0$. By exactness of the SES we get $\sum n_{it} b_i + n_t b_t = f(a_t)$ for some $a_t \in A$. We show that the elements a_t for $t = s+1, ..., r$ are independent in A. Suppose $\sum_{t=s+1,...,r} m_t a_t = 0$, then $\sum m_t f(a_t) = \sum m_t (\sum n_{it} b_i + n_t b_t) = 0$, a relation among the $b_i' s$. The coefficient of b_t is $m_t n_t$ and must be 0. Since $n_t \neq 0, m_t = 0$ and the relation was trivial. Thus $\mathrm{rk}(B) \leq \mathrm{rk}(A) + \mathrm{rk}(C)$.

*Exercise 4: *Chain Complexes and their Homology*

Let $C = (C_*, \delta_*)$ be a chain complex of abelian groups. In this exercise and often during class, for $m \in \mathbb{N}$ and a group G, we will let

$$G^{\oplus m} = \bigoplus_{i=1}^{m} G.$$

Furthermore, if $G = \langle a \rangle$, hence is generated by one element (infinite or finite), and if $1 \leq i \leq m$, we will let $e_i \in G^{\oplus m}$ denote the element $(0, \ldots, a, \ldots, 0) \in G^m$, where a is the i-th entry.

(a) Assume $G = \langle a \rangle$. Show that $G^{\oplus m} = \langle e_1, e_2, \ldots, e_m \rangle$.

(b) Assume C is a *long exact sequence (LES)*, that is, assume $\operatorname{im} \delta_n = \ker \delta_{n-1}$ for all $n \in \mathbb{Z}$. Show that the homology of C is trivial, that is, $H_n(C) = \{0\}$ for all $n \in \mathbb{Z}$.

(c) Given the following chain complex

$$C: \ \ldots \longrightarrow 0 \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_0} 0 \longrightarrow \ldots$$

where $\delta_1(e_1) = e_2 - e_1$, $\delta_1(e_2) = e_3 - e_2$, and $\delta_1(e_3) = e_1 - e_3$. Calculate the homology $H_*(C)$.

(d) Given the following chain complex

$$D: \ \ldots \longrightarrow 0 \xrightarrow{\delta_3} \mathbb{Z}^{\oplus 2} \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\delta_0} 0 \longrightarrow \ldots$$

where $\delta_1 = 0$ and $\delta_2(e_i) = e_1 + e_2 + e_3$ for $i = 1, 2$. Calculate the homology $H_*(D)$.

Solution:

Let G be a group generated by the element a and let $e_i := (0, ..., 0, a, 0, ..., 0)$ denote the element of $G^{\oplus m}$ with a in the i-th coordinate for $1 \leq i \leq m$.

(a) Let $(x_1, ..., x_m) \in G^{\oplus m}$ then $x_i \in G$ for all $i = 1, ..., m$. Hence there exist $\alpha_1, ..., \alpha_m \in \mathbb{Z}$ such that $x_i = \alpha_i a$ for all $i = 1, ..., m$. Thus, $(x_1, ..., x_m) = \sum_{i=1}^{m} \alpha_i e_i$. Since $e_i \in G^{\oplus m}$ for $i = 1, ..., m$, the reverse inclusion holds trivially.

(b) First note that a LES is a chain complex. Let $\mathcal{C}$ be a LES, then $H_n(\mathcal{C}) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1}) = 0$ for all $n \in \mathbb{Z}$.

(c) For $H_0(\mathcal{C})$ we have $\ker(\partial_0) = \mathbb{Z}^{\oplus 3}$, since ∂_0 is the zero-map. The image

$$\begin{aligned}
\operatorname{im}(\partial_1) &=< e_2 - e_1, e_3 - e_2, e_1 - e_3 > \\
&=< (e_2 - e_1) + (e_3 - e_2) + (e_1 - e_3), e_3 - e_2, e_1 - e_3 > \\
&=< e_3 - e_2, e_1 - e_3 > \\
&=< a, b >
\end{aligned}$$

If we remove the generators accordingly. Hence $\operatorname{im}(\partial_1) \simeq \mathbb{Z}^{\oplus 2}$ and so we get $H_0(\mathcal{C}) \simeq \mathbb{Z}^{\oplus 3}/\mathbb{Z}^{\oplus 2} \simeq \mathbb{Z}$.

For $H_1(\mathcal{C})$ we have $\ker(\partial_1) =< e_1 + e_2 + e_3 >$ and $\operatorname{im}(\partial_2) = 0$. Hence $H_1(\mathcal{C}) \simeq \mathbb{Z}$.

For $H_k(\mathcal{C})$ for $k \neq 0, 1$: Since $\ker(\partial_k) = 0$ for all $k \neq 0, 1$ we get $H_k(\mathcal{C}) = 0$ for $k \neq 0, 1$.

(d) Looking at δ_0, we get $\operatorname{im}(\delta_0) = 0$ and $\ker(\delta_0) = \mathbb{Z}$ since δ_0 is the zero-map. Looking at δ_1, we get $\operatorname{im}(\delta_1) = 0$ and $\ker(\delta_1) = \mathbb{Z}^{\oplus 3}$ since δ_1 is the zero-map.

Looking at δ_2, we get $\operatorname{im}(\delta_2) =< (1,1,1) >\simeq \mathbb{Z}$ since $\delta_2(1,0) = \delta_2(0,1) = (1,1,1)$ and obviously $\ker(\delta_2) =< (1,-1) >\simeq \mathbb{Z}$. Together with $\operatorname{im}(\delta_k) = \ker(\delta_k) = 0$ for $k \geq 3$ we can compute all $H_*(D)$ and get

$$\begin{aligned}
H_0(D) &= \ker(\delta_0)/\operatorname{im}(\delta_1) = \mathbb{Z}/0 = \mathbb{Z} \\
H_1(D) &= \ker(\delta_1)/\operatorname{im}(\delta_2) \simeq \mathbb{Z}^{\oplus 3}/\mathbb{Z} \simeq \mathbb{Z}^{\oplus 2} \\
H_2(D) &= \ker(\delta_2)/\operatorname{im}(\delta_3) \simeq \mathbb{Z}/0 = \mathbb{Z} \\
H_k(D) &= \ker(\delta_k)/\operatorname{im}(\delta_{k+1}) = 0/0 = 0 \qquad \forall k \geq 3.
\end{aligned}$$