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Topologie II – Exercise Sheet 2

Date of assignment: **Monday, Oct. 28, 2014**. We highly recommend problems marked with a star. Do the other exercises if they seem challenging enough or if you don't have an idea of how to solve them immediately.

Exercise 1: *Chain Homotopy*

Show that "chain homotopy" of chain maps is an equivalence relation. Specify on which set (or class or category) you are defining this equivalence relation.

*Exercise 2: *Short Exact Sequences and Ranks*

Let $\mathcal{C}$ be an abelian category. An exact sequence is called a *short exact sequence (SES)* if it is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where 0 is the *zero object* and A, B, C are objects in $\mathcal{C}$ such that the morphisms as indicated have the property that the image of one morphism is equal to the kernel of the next morhpism. Since the zero object is both *terminal* and *initial*, the morphisms from 0 to A and from C to 0 are uniquely determined by A and C.

(a) Show that f is injective (define this first).

(b) Show that g is surjective (define this first).

(c) Assume only for part (c) that C is equal to the zero object. Show that f is an isomorphism (define this first). Notation: $A \cong B$ via f.

Assume for the remaining exercise that $\mathcal{C} = \mathbf{Ab}$ (the category of abelian groups).

(d) Show that $\operatorname{im} f \cong A$ and that $B/\operatorname{im} f \cong C$.

(e) Define the *rank* $\operatorname{rk} G$ of an abelian group G as the cardinality of a maximally $\mathbb{Z}$-linearly independet subset (this is well defined!). Assume that A, B, C have finite rank. Prove that $\operatorname{rk} B = \operatorname{rk} A + \operatorname{rk} C$. Draw an analogy to the dimension formulas for linear maps between vector spaces and the quotient vector space!

Exercise 3: *Short Exact Sequences: Examples*

(a) Determine whether there exists a SES

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/8 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow 0.$$

(b) For $n \in \mathbb{N}$, determine which abelian groups G fit into the SES

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z}/n \longrightarrow 0.$$

***Exercise 4:** *Chain Complexes and their Homology*

Let $C = (C_*, \delta_*)$ be a chain complex of abelian groups. In this exercise and often during class, for $m \in \mathbb{N}$ and a group G, we will let

$$G^{\oplus m} = \bigoplus_{i=1}^{m} G.$$

Furthermore, if $G = \langle a \rangle$, hence is generated by one element (infinite or finite), and if $1 \leq i \leq m$, we will let $e_i \in G^{\oplus m}$ denote the element $(0, \ldots, a, \ldots, 0) \in G^m$, where a is the i-th entry.

(a) Assume $G = \langle a \rangle$. Show that $G^{\oplus m} = \langle e_1, e_2, \ldots, e_m \rangle$.

(b) Assume C is a *long exact sequence (LES)*, that is, assume $\operatorname{im} \delta_n = \ker \delta_{n-1}$ for all $n \in \mathbb{Z}$. Show that the homology of C is trivial, that is, $H_n(C) = \{0\}$ for all $n \in \mathbb{Z}$.

(c) Given the following chain complex

$$C: \ \ldots \longrightarrow 0 \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_1} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_0} 0 \longrightarrow \ldots$$

where $\delta_1(e_1) = e_2 - e_1$, $\delta_1(e_2) = e_3 - e_2$, and $\delta_1(e_3) = e_1 - e_3$. Calculate the homology $H_*(C)$.

(d) Given the following chain complex

$$D: \ \ldots \longrightarrow 0 \xrightarrow{\delta_3} \mathbb{Z}^{\oplus 2} \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 3} \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\delta_0} 0 \longrightarrow \ldots$$

where $\delta_1 = 0$ and $\delta_2(e_i) = e_1 + e_2 + e_3$ for $i = 1, 2$. Calculate the homology $H_*(D)$.