Berlin Mathematical School
Freie Universität Berlin

Prof. Pavle Blagojević
Albert Haase
Prof. Holger Reich

Arbeitsgruppe Diskrete Geometrie

Arbeitsgruppe Algebraische Topologie

Topologie II – Exercise Sheet 3

Date of assignment: **Monday, Nov. 17, 2014**. We highly recommend problems marked with a star.

Exercise 1: *Short Exact Sequence That Does Not Split*

Given the abelian groups $\mathbb{Z}$, $\mathbb{Z}\oplus(\mathbb{Z}/2)^{\mathbb{N}}$ and $(\mathbb{Z}/2)^{\mathbb{N}}$ construct a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with these groups such that $B \cong A \oplus C$ and the sequence does not split.

*Exercise 2: *Homology of the Suspension*

Given a topological space X we define the *suspension* SX of X as the quotient space

$$SX := X \times [0,1]/\sim$$

where $\sim$ is the equivalence relation generated by: $(x,s) \sim (y,t)$ if and only if $s = t = 0$ or $s = t = 1$. Show that for the reduced homology groups

$$\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX) \quad \text{for all } n.$$

*Exercise 3: *Homology of Complements*

(a) Suppose U and V are open sets in $\mathbb{R}^d$ and $H_n(U \cup V) = 0$ for all $n \geq 1$. Show that $H_n(U \cap V) \cong H_n(U) \oplus H_n(V)$ for all $n \geq 1$.

(b) Suppose A and B are disjoint closed sets in $\mathbb{R}^d$. Show that

$$H_n(\mathbb{R}^d \setminus (A \cup B)) \cong H_n(\mathbb{R}^d \setminus A) \oplus H_n(\mathbb{R}^d \setminus B) \quad \text{for all } n \geq 1.$$

What can be said for H_0?

(c) Let U be an open subset of $\mathbb{R}^n$ and let $K \subset U$ be compact. Show that

$$H_n(U \setminus K) = H_n(U) \oplus H_n(\mathbb{R}^2 \setminus K) \quad \text{for all } n \geq 1.$$

***Exercise 4:** *Homology of the Wedge of two Spaces*

Given topological spaces X and Y and "base points" $x_0 \in X$ and $y_0 \in Y$, the *wedge* of X and Y (with respect to the base points) is defined as the quotient space

$$X \vee Y := X \sqcup Y / \sim$$

where $\sim$ is the equivalence relation generated by $x_0 \sim y_0$. Assume that x_0 is a deformation retract of an open set $U \subseteq X$ and y_0 is a deformation retract of an open set $V \subseteq Y$. Show that

$$\widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \quad \text{for all } n.$$