Berlin Mathematical School

Freie Universität Berlin

Prof. Pavle Blagojević
Albert Haase
Prof. Holger Reich

Arbeitsgruppe Diskrete Geometrie

Arbeitsgruppe Algebraische Topologie

Topologie II – Exercise Sheet 3

Exercise 1: *Short Exact Sequence Does Not Split*

Given the abelian groups $\mathbb{Z}$, $\mathbb{Z}\oplus(\mathbb{Z}/2)^{\mathbb{N}}$ and $(\mathbb{Z}/2)^{\mathbb{N}}$ construct a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with these groups such that $B \cong A \oplus C$ and it does not split.

Solution to Exercise 1:

Cosinder the sequence

$$0 \overset{i}{\underset{s}{\rightleftarrows}} \mathbb{Z} \oplus (\mathbb{Z}/2)^{\mathbb{N}} \overset{j}{\longrightarrow} (\mathbb{Z}/2)^{\mathbb{N}} \longrightarrow 0 \qquad (*)$$

where $i(z) := (2z, 0)$ and $\mathrm{im}(i) = 2\mathbb{Z} \oplus 0$.
We have $\ker(j) = 2\mathbb{Z} \oplus 0$ and $\mathrm{im}(j) = (\mathbb{Z}/2)^{\mathbb{N}}$.
Hence the sequence $(*)$ is exact. Assume $(*)$ splits. Then there is a homomorphism $s : \mathbb{Z} \oplus (\mathbb{Z}/2)^{\mathbb{N}} \longrightarrow \mathbb{Z}$ such that $s \circ i = id_{\mathbb{Z}}$. Then every element $a \in (\mathbb{Z}/2)^{\mathbb{N}}$ must be mapped to 0 by s, since s preserves the order of elements (and a has finite order). Then s is given by $m \in \mathbb{Z}$ such that $s(z, a) = mz$. Hence $s \circ i(z) = m2z \neq z$ leading to a contradiction.

*Exercise 2: *Homology of the Suspension*

Given a topological space X we define the *suspension* SX of X as

$$SX := X \times [0, 1]/\sim$$

where $\sim$ is the equivalence relation generated by: $(x, s) \sim (y, t)$ if and only if $s = t = 0$ or $s = t = 1$. Show that

$$\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX) \quad \text{for all } n.$$

Solution to Exercise 2:
we will show that $\widetilde{H}_n(X) = \widetilde{H}_{n+1}(SX)$ by applying the Mayer-Vietoris sequence for reduced homology to the following sets:

$$U := X \times \left[\tfrac{1}{4}, 1\right] / \sim$$
$$V := X \times \left[0, \tfrac{3}{4}\right] / \sim$$

Note that $SX = \mathrm{int}(U) \cup \mathrm{int}(V)$. We will show that:
(1) U and V can each be deformation retracted to a point.
(2) $U \cap V$ can be deformation retracted to X.
Then the Mayer-Vietoris sequence in reduced homology is for $n \in \mathbb{N}_{\geq 0}$:

$$\longrightarrow 0 \oplus 0 \longrightarrow \widetilde{H}_{n+1}(SX) \longrightarrow \widetilde{H}_n(X) \longrightarrow 0 \oplus 0 \longrightarrow$$

This implies that $\widetilde{H}_{n+1}(SX) \cong \widetilde{H}_n(X)$ for all $n \in \mathbb{N}_{\geq 0}$.
We will prove (1) for V. The proof for U is analogous. Define a map:

$$F : X \times \left[0, \tfrac{3}{4}\right] \times [0,1] \longrightarrow X \times \left[0, \tfrac{3}{4}\right] / \sim = V$$
$$(x, s, t) \longmapsto [(x, (1-t)s)]$$

This is continuous since it is the composition of continuous maps.
Let $\pi : X \times \left[0, \frac{3}{4}\right] \times [0,1] \longrightarrow V \times [0,1]$ be the quotient map. Then the map

$$\widetilde{F} : V \times [0,1] \longrightarrow V$$
$$([(x,s)], t) \longmapsto [(x, (1-t)s)]$$

is well defined an continuous because $\widetilde{F} \circ \pi = F$ and F is continuous. (Notice that $\widetilde{F}$ is not well defined if we take $V = X \times [0,1] / \sim$.)
Now we check that $\widetilde{F}$ is a deformation retraction to the point $[(x, 0)]$:
(a) $F([(x,s)], 0) = [(x,s)]$ for all $[(x,s)] \in V$.
(b) $F([(x,s)], 1) = [(x,0)]$ for all $[(x,s)] \in V$
(c) $F([(x,0)], 1) = [(x,0)]$ for all $x \in X$.
To prove (2) note that $U \cap V \cong X \times [\frac{1}{4}, \frac{3}{4}]$. Define

$$G : X \times [\tfrac{1}{4}, \tfrac{3}{4}] \times [0,1] \longrightarrow X \times [\tfrac{1}{4}, \tfrac{3}{4}]$$
$$(x, s, t) \longmapsto (x, (1-t)(s - \tfrac{1}{4}) + \tfrac{1}{4}).$$

Then G is continuous and a deformation retraction to $X \times \{\frac{1}{4}\} \cong X$.

*Exercise 3: *Homology of Complements*

(a) Suppose U and V are open sets in $\mathbb{R}^d$ and $H_n(U \cup V) = 0$ for all $n \geq 1$. Show that $H_n(U \cap V) \cong H_n(U) \oplus H_n(V)$ for all $n \geq 1$.

(b) Suppose A and B are disjoint closed sets in $\mathbb{R}^d$. Show that

$$H_n(\mathbb{R}^d \setminus (A \cup B)) \cong H_n(\mathbb{R}^d \setminus A) \oplus H_n(\mathbb{R}^d \setminus B) \quad \text{for all } n \geq 1.$$

What can be said for H_0?

(c) Let U be an open subset of $\mathbb{R}^n$ and let $K \subset U$ be compact. Show that

$$H_n(U \setminus K) = H_n(U) \oplus H_n(\mathbb{R}^2 \setminus K) \quad \text{for all } n \geq 1.$$

Solution to Exercise 3:

These exercises are applications of the Mayer-Vietoris sequence. Let us recall the statement:

Let X be topological space and $U, V \subseteq X$ subspaces such that $\mathrm{int}(U) \cup \mathrm{int}(V) = X$. Then the following sequence of (reduced) homology groups is exact:

$$\ldots \longrightarrow H_{n+1}(X) \xrightarrow{\partial_*} H_n(U \cap V) \xrightarrow{(i_*, j_*)} H_n(U) \oplus H_n(V) \xrightarrow{u_* - l_*} H_n(X) \longrightarrow \ldots$$

(a) Set $X := U \cup V$ and plug $H_n(X) = 0$ into the sequence.

(b) Set $X = \mathbb{R}^d$ and $U := \mathbb{R}^d \setminus A$ and $V := \mathbb{R}^d \setminus B$. Then $\mathrm{int}(U) = U$ and $\mathrm{int}(V) = V$ and $U \cup V = \mathbb{R}^d \setminus (A \cap B) = \mathbb{R}^d$. Then $H_n(X) = 0$ for all $n \geq 1$. Plug this into the sequence to get the desired isomorphism.
For H_0 the analogous statement is not true. For a counterexample take $A, B \subseteq \mathbb{R}^2$ to be two distinct points. Then $H_0(\mathbb{R}^2 \setminus (A \cup B)) \cong \mathbb{Z}$ and $H_0(\mathbb{R}^2 \setminus A) \oplus H_0(\mathbb{R}^2 \setminus B) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(c) Set $V := \mathbb{R}^2 \setminus K$. Then V is open because K is closed as a compact set. Hence $\mathbb{R}^2 = U \cup V = \mathrm{int}(U) \cup \mathrm{int}(V)$. As in (b) $H_n(\mathbb{R}^2) = 0$ for all $n \geq 1$. If we plug this into the sequence we get the desired result.

*Exercise 4: *Homology of the Wedge of two Spaces*

Given topological spaces X and Y and "base points" $x_0 \in X$ and $y_0 \in Y$, the *wedge* of X and Y is definded as

$$X \vee Y := X \sqcup Y / \sim$$

where $\sim$ is the equivalence relation generated by $x_0 \sim y_0$. Assume that x_0 is a deformation retract of an open set $U \subseteq X$ and y_0 is a deformation retract of an open set $V \subseteq Y$. Show that

$$\widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \quad \text{for all } n.$$

Solutions to Exercise 4:

We will proceed similarly as in Exercise 2. We will define sets $\widetilde{U}$ and $\widetilde{V}$ such that

(1) $\operatorname{int}(\widetilde{U}) \cup \operatorname{int}(\widetilde{V})$ deformation retracts to $X \vee Y$.

(2) $\widetilde{U}$ and $\widetilde{V}$ deformation retract to homeomorphic copies of X respectively Y.

(3) $\widetilde{U} \cap \widetilde{V}$ deformation retracts to a point.

Then the Mayer-Vietoris sequence will give the desired isomorphism $\widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \cong \widetilde{H}_n(X \vee Y)$ for all $n \in \mathbb{N}_{\geq 0}$.

Define $\widetilde{U} := U \times Y/\sim$ and $\widetilde{V} := X \times V/\sim$.

To show (2), let $F_1 : U \times [0,1] \longrightarrow U$ be the deformation retraction to x_0 and $F_2 : V \times [0,1] \longrightarrow V$ be the deformation retraction to y_0 then define:

$$\begin{aligned}
\widetilde{F_1} : U \times Y \times [0,1] &\longrightarrow U \times Y \\
(u, y, t) &\longmapsto (F_1(u,t), y) \\
\widetilde{F_2} : X \times V \times [0,1] &\longrightarrow X \times V \\
(x, v, t) &\longmapsto (x, F_2(v,t)).
\end{aligned}$$

If we now pass to the maps on the quotients, we get deformation retractions that send $\widetilde{U}$ to $\{x_0\} \times Y \approx Y$ and $\widetilde{V}$ to $X \times \{y_0\} \approx X$. (1) and (3) are shown in a similar way as (2).