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Topologie II – Exercise Sheet 4

*Exercise 1: *Deformation Retractions*

(a) Show that if A is a retract of X, meaning there exists a retraction $r\colon X \longrightarrow A$, then the maps

$$H_n(A) \longrightarrow H_n(X)$$

in homology induced by the inclusion $A \hookrightarrow X$ are injective.

(b) Give an example of a space that is contractible but does not deformation retract to a point.

(c) The following topological space, up to homeomorphism, is called the *Möbius strip*:

$$M := [0,1] \times [0,1]/\sim ,$$

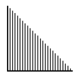
where $\sim$ is the equivalence relation generated by $(0,t) \sim (1, 1-t)$. Show that M deformation retracts to a circle. By circle we mean a space homeomorphic to S^1.

Solution to Exercise 1:

(a) Let $r : X \longrightarrow A$ be a retraction, that is, $A \subseteq X$ and $r_{|A} = id_A$. Hence, if $i : A \hookrightarrow X$ denotes the inclusion map, then $r \circ i = id_A$. Let r_*, i_*, id_{A*} denote the maps on the level of homology. Then by functorality (because homology is a functor) we get $r_* \circ i_* = id_{A*}$. This directly implies that i_* is injective.

(b) The wording of this exercise was not very precise. Here the difference between "deformation retraction" and "strong deformation retraction" is paramount.
A *strong deformation retraction* is a deformation retraction $F : X \times [0,1] \longrightarrow X$ of a topological space X to a subspace $A \subseteq X$ that leaves A fixed for all times $t \in [0,1]$. More precisely $F(a,t) = a \ \forall a \in A, t \in [0,1]$. It is an easy exercise to verify that a space is contractible if and only if it deformation retracts to a point (in the weaker sense). Hence the correct wording of this exercise is:

Give an example of a space X, that is contractible but does not strong deformation retract to a point.
The following example is slightly involved and is taken from Hatcher[1]. We will make use of the following property of spaces X that strong deformation retract to a point: If a space X has a strong deformation retract to a point $x_0 \in X$, then for every neighborhood U of x_0 there exists a neighborhood V of x_0 contained in U such that the inclusion $i : V \hookrightarrow U$ is homotopic to a constant map. Note that such a neighborhood V must be path connected and that this property is not true for deformation retractions. Consider the following space $X := [0,1] \times \{0\} \cup \bigcup_{r \in \mathbb{Q} \cap [0,1]} \{r\} \times [0, 1-r]$ considered as a subspace of $\mathbb{R}^2$.



We can easily show that X strong deformation retracts to $[0,1] \times \{0\}$ and further that X strong deformation retracts to any point $x_0 \in [0,1] \times \{0\}$. Since no neighborhood of any point $x_0 \in X \setminus [0,1] \times \{0\}$ is path connected, X cannot strong deformation retract to any such point.

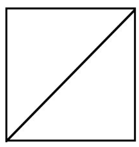
Now consider the following space Y obtained by gluing copies of X together in the pictured way.

$\cdots$  $\cdots$

Again we consider Y as an subspace of $\mathbb{R}^2$. Observe that no point in Y has a path connected neighborhood, hence Y cannot strong deformation retract to any point. One now shows that Y is homotopic to the thick zigzag-line, which is homotopic to $\mathbb{R}$, hence Y is contractible.

□

(c) Consider the following map $F : [0,1] \times [0,1] \times [0,1] \longrightarrow [0,1] \times [0,1]$ with $F(r,s,t) := (r, tr + (1-t)s)$ and consider the following figure



F is a deformation retract to the diagonal given by $C := \{(r,r) \mid r \in [0,1]\}$. Now we pass to the quotient M and consider the map $\widetilde{F} : M \times [0,1] \longrightarrow M$ where $\widetilde{F}([(r,s)],t) := [F(r,s,t)]$ which is continuous. It is easy to verify that $widetildeF$ is a deformation retraction of M to $C/\sim$. Also, the space

[1] Allen Hatcher: *Algebraic Topology*, page 18, exercise 6 (b).

$C/\sim$ is homeomorphic to a circle (the endpoints of C are identified under the equivalence relation).

Exercise 2: *Open Maps*

Show that a continuous *and bijective* map from a compact space to a Hausdorff space is open, that is, sends open sets to open sets.

***Exercise 3:** *Topology of Simplicial Complexes*

(a) Give an example of a geometric simplicial complex in some $\mathbb{R}^d$ whose topology does not agree with the subspace topology.

(b) Show that a simplicial complex is a Hausdorff space.

(c) Show that a simplicial complex is compact if and only if it is finite, meaning it is a set of only finitely many simplices.

Solution to Exercise 3:

(a) Consider the following 0-dimensional simplicial complex:
$K := \{\{\frac{1}{n}\}, \{0\} \mid n \in \mathbb{N}\}$.
Then $\{0\}$ is open in $|K|$ because the intersection with every set $\{\frac{1}{n}\}, \{0\}$ is open in the set (either this is the empty set or it is $\{0\}$). As a subspace of $\mathbb{R}$ however, $\{0\} \subseteq |K|$ is not open because there exists no open set $U \subseteq \mathbb{R}$ such that $U \cap |K| = \{0\}$.

(b) The proof is due Munkres, Lemma 2.4.
Given a topological space X and a simplicial complex K, a map $f : |K| \longrightarrow X$ is continuous if and only if $f_{|\sigma} : \sigma \longrightarrow X$ is continuous for all simplices $\sigma \in K$. If v is a vertex of K and $x \in |K|$ a point and $x = \sum t_i a_i$ is a convex combination of the unique simplex that contains x in its relative interior, then we define $t_v(x) = t_i$ if $v = a_i$ for some i and $t_v(x) = 0$ otherwise. This defines a function $t_v : \text{conv}(a_i) \longrightarrow \mathbb{R}$ that is continuous on $\text{conv}(a_i)$ because it is constant. Hence t_v is continuous on $|K|$ by the statement above.
Given two point $x, y \in |K|$, there is at least one vertex $v \in K$ such that $t_v(x) \neq t_v(y)$. Choose r in the open interval $(t_v(x), t_v(y))$ (or $(t_v(y), t_v(x))$). Now the sets $\{x \mid t_v(x) < r\}$ and $\{x \mid t_v(x) > r\}$ are open disjoint neighborhoods containing x and y.

□

(c) If K is finite it is compact or the union of finitely many compact sets. If K is not finite, take an ε-neighborhood for every simplex σ in K and cover $|K|$ with these neighborhoods. This cover will not have a finite subcover.

□

***Exercise 4:** *Simplicial Homology*

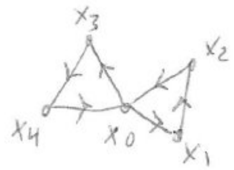
(a) Compute the simplicial homology of a wedge of two circles directly. Extend your reasoning and model to compute the simplicial homology of the wedge of two spheres $S^k \vee S^\ell$.

(b) Calculate the simplicial homology of the Möbius strip M directly. *Note:* Not every triagulation of the square will lead to a triangulation of M!

(c) How could we have calculated the homology of M more easily?

Solution to Exercise 4:

(a) We will use tha following simplicial model for $\mathbb{S}^1 \vee \mathbb{S}^1$

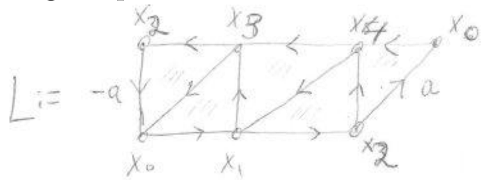


The simplicial complex K is homeomorphic to $\mathbb{S}^1 \vee \mathbb{S}^1$ via "radial projection" with two centers. By counting faces of K we have $C_0(K) \cong \mathbb{Z}^{\oplus 5}, C_1(K) \cong \mathbb{Z}^{\oplus 6}$ and $C_i(K) \cong 0$ for all $i \geq 2$.
Since $|K|$ is path connected $H^0(K) \cong \mathbb{Z}$. The group $H^j(K) = 0$ for $j \geq 2$ since there are no faces of dimension higher than 1. For $H^1(K)$ we compute:

$$n_0[x_0, x_1] + n_1[x_1, x_2] - n_2[x_0, x_3] = n_3[x_0, x_1] + n_4[x_3, x_4] - n_5[x_0, x_4] \in \ker \partial_1$$

Then $-n_0x_0 + (n_0 - n_1)x_1 = (n_1 - n_2)x_2 + (n_2 - n_3)x_0 + (n_3 - n_4)x_3 + (n_4 - n_5)x_4 + n_5x_4 = 0$. Implying that $(n_2 - n_3) = (n_0 - n_5)$ and
$(n_0 - n_1) = (n_1 - n_2) = (n_3 - n_4) = (n_4 - n_5) = 0$. Hence $n_1 = n_2 = n_0$ and $n_3 = n_4 = n_5$ which implies that $H^1(K)$ is generated by $([x_0, x_1] + [x_1, x_2] - [x_0, x_2])$ and $([x_0, x_3] = [x_3, x_4] - [x_0, x_4])$ and hence is isomorphic to $\mathbb{Z}^{\oplus 2}$. On the last sheet we showed that the homology of $\mathbb{S}^k \vee \mathbb{S}^\ell$ is the direct product of the homology of $\mathbb{S}^k$ and the homology of S^l since we can form the wedge sum at points that are deformation retracts of open neighborhoods in $\mathbb{S}^k$ respectively $\mathbb{S}^l$. In this exercise we can verify this by taking the join of the boundary complexes of a $(k + 1)$ and an $(l + 1)$ simplex attached at a vertex.

(b) We will use the following simplicial model for M:



Clearly $|L| \approx M$. By counting faces we get:
$C_0(L) \cong \mathbb{Z}^{\oplus 5}, C_1(L) \cong \mathbb{Z}^{\oplus 10}, C_2(L) \cong \mathbb{Z}^{\oplus 5}, C_j(L) \cong 0$ for all $j \geq 3$.Since M is

path connected (images of path connected spaces are path connected) we know that $H^0(L) = \mathbb{Z}$. Thus we get: $H^2(L) = 0$ by calculating $\ker \partial_2$ and $H^1(L) = \mathbb{Z}$ by calculating $\ker \partial_1 / \operatorname{im} \partial_2$.

(c) In Exercise 1 (c) we showed that M has a deformation retract to a circle, hence it must have the homology of a circle.