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Topologie II – Exercise Sheet 5

Date of assignment: **Thursday, Nov. 27, 2014**. We highly recommend problems marked with a star.

*Exercise 1: *Relative Homology*

Let (X, A) be a pair of spaces, that is, X is a topological space and $A \subseteq X$.

(a) Show that the inclusion $A \hookrightarrow X$ induces induces isomorphisms on homology groups if and only if $H_n(X, A) = 0$ for all n.

(b) Calculate the homology of an n-sphere by choosing X as the simplicial complex of all faces of an $(n+1)$-simplex and A as the subcomplex of all proper faces, that is, all faces $F \in X$ such that F is not the $(n+1)$-simplex.

(c) Show that $H_0(X, A) = 0$ if and only if A intersects every path-component of X.

(d) Show that $H_1(X, A) = 0$ if and only if $H_1(A) \longrightarrow H_1(X)$ is surjective and each path component of X contains at most one path component of A.

(e) Let $X = S^2$ and A be a finite set of points in X. Calculate $H_*(X, A)$.

(f) Let $X = \mathbb{R}$ and $A = \mathbb{Q} \subset \mathbb{R}$ be the set of rational numbers. Show that $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian.

*Exercise 2: *Maps and Homotopy Equivalences of Pairs of Spaces*

Given pairs of spaces (X, A) and (Y, B) a *map of pairs (of spaces)* $f\colon (X, A) \longrightarrow (Y, B)$ is a continuous map $f\colon X \longrightarrow Y$ such that $f(A) \subseteq B$. The identity $\mathrm{id}_{(X,A)}\colon (X, A) \longrightarrow (X, A)$ is the identity id_X. A *homotopy equivalence of two maps of pairs* $f, f'\colon (X, A) \longrightarrow (Y, B)$, denoted by $f \simeq f'$, is given by a continuous map $F\colon X \times [0, 1] \longrightarrow Y$ such that $F(a, t) \in B$ for all $a \in A$ and all $t \in [0, 1]$ and $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for all $x \in X$. A map $f\colon (X, A) \longrightarrow (Y, B)$ of pairs of spaces is called *homotopy equivalence (of pairs)* if there exists a map of pairs $g\colon (Y, B) \longrightarrow (X, A)$ such that $f \circ g \simeq \mathrm{id}_{(Y,B)}$ and $g \circ f \simeq \mathrm{id}_{(X,A)}$. If $A = B = \emptyset$ then we are in the case of continuous maps $X \longrightarrow Y$ and we drop the "of pairs" in our definitions.

Let $f, f' \colon (X, A) \longrightarrow (Y, B)$ be maps of pairs.

(a) Let $F \colon f \simeq f'$ be a homotopy of pairs. Show that F induces a homotopy between $f_{|A}$ and $f'_{|A}$.

(b) Assume now that $f \colon X \longrightarrow Y$ and $g \colon A \longrightarrow B$, $g(x) = f(x)$ are both homotopy equivalences. Show that $f_* \colon H_n(X, A) \longrightarrow H_n(Y, B)$ is an isomorphism for all $n \geq 0$.

(c) Let f be the "inclusion" $(D^n, S^{n-1}) \hookrightarrow (D^n, D^n \setminus \{0\})$, where D^n denotes the n-dimensional disc and S^{n-1} denotes its boundary. Show that f is not a homotopy equivalence of pairs. Hence the hypotheses of (b) do not imply that f is a homotopy equivalence of pairs.

Exercise 3: *Quotients*

(a) The real projective plane $\mathbb{R}P^2$ can be constructed as a quotient of the 2-disc by dividing the boundary circle of the 2-disc into two semicircles and identifying them with opposite orientations. Show that removing a closed disc from $\mathbb{R}P^2$ yields a Möbius strip. Construct $\mathbb{R}P^2$ from the unit square by identifying edges.

(b) Given a solid oriented triangle $\Delta = [v_0, v_1, v_2]$ in $\mathbb{R}^2$, form a quotient of Δ by identifying the two edges $[v_0, v_1]$ and $[v_1, v_2]$. What is the resulting topological space (up to homeomorphism)?

(c) Let $\Delta = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}$ and let $\sim$ be the equivalence relation generated by $(x, 0) \sim (0, x)$ and $(x, 0) \sim (x, 1 - x)$. Show that D is *contractible*, that is, homotopy equivalent to a point. Construct a triangulation of D.