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Topologie II – Exercise Sheet 6

Date of assignment: **Monday, Dec. 22, 2014**. We highly recommend problems marked with a star.

*Exercise 1: *Definition of Degree*

Recall the definition of *degree* of a continuous map $f\colon S^n \longrightarrow S^n$ for $n \in \mathbb{Z}_{\geq 0}$ as the unique integer $\deg(f) \in \mathbb{Z}$ such that $f_*(x) = \deg(f)x$ for all $x \in H_n(S^n)$. In this exercise we complete the proof of a proposition from the tutorial. Prove the statements in red:

(a) If $f = \mathrm{id}_{S^n}$ then, $\deg(f) = 1$.
(b) If f is not surjective, then $\deg(f) = 0$.
(c) Homotopic maps have the same degree.
(d) If $g\colon S^n \longrightarrow S^n$ is another continuous map, then $\deg(f \circ g) = \deg(f)\deg(g)$. In particular homotopy equivalences have degree -1 or $+1$.
(e) A reflection in a (linear) coordinate hyperplane has degree -1.
(f) The antipodal map $a\colon S^n \longrightarrow S^n$ given by $a(x) = -x$ has degree $(-1)^{n+1}$.
(g) If f has no fixed points, then it has degree $(-1)^{n+1}$.

*Exercise 2: *Degrees and free $\mathbb{Z}/2$-Actions*

A *group action* of a group G on a topological space X is a group homomorphism $G \longrightarrow \mathrm{Homeo}(X)$, where the codomain is the set of all homeomorphisms on X. By abuse of notation, we often denote the image of $g \in G$ under this homomorphism by g as well. A space with a G-action is called a *G-space*. A group action is called *free* if for every $g \in G, g \neq e$ the homeomorphism $g\colon X \longrightarrow X$ has no fixed points. A map $f : X \longrightarrow Y$ for two G-spaces X, Y is called *G-equivariant* if $f(gx) = gf(x)$ for all $g \in G$ and all $x \in X$. Given an n-sphere S^n, we call the action of $\mathbb{Z}/2 = \{-1, 1\}$ on S^n given by $1x = x$ and $-1x = -x$ for all $x \in S^n$ the *antipodal action*.

(a) Prove or disprove: The antipodal action is free.

(b) Show that $\mathbb{Z}/2$ is the only nontrivial group that can act freely on S^n for even n. What does this imply for a rotation of the 2-sphere (well known result)?

(c) Assume the following non-trivial theorem: Any continuous map $f\colon S^n \longrightarrow S^n$ that is equivariant with respect to the antipodal action has odd degree. Prove that there does *not* exist a continuous map $f\colon S^{n+1} \longrightarrow S^n$ that is equivariant with respect to the antipodal action.

(d) Show that the second statement in (b) is equivalent to the following statement: For any continuous map $f\colon S^{n+1} \longrightarrow \mathbb{R}^n$ there exists a point $x \in S^{n+1}$ such that $f(x) = f(-x)$.

(e) Deduce the Brouwer fixed point theorem from (c) or (d).

Merry Christmas!