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Topologie II – Exercise Sheet 7

Date of assignment: **Monday, Jan. 26, 2015**. We highly recommend problems marked with a star.

Exercise 1: *Homology of a non-orientable surface of genus g*

A *non-orientable surface of genus g, called M'_g*, is a closed, connected, non-orientable 2-dimensional, smooth real manifold that is obtained by gluing $g \geq 1$ cross-caps to a 2-sphere S^2. The *gluing of* 1 *cross-cap* is the following contruction: Take S^2 and cut out an open 2-disk. Then identify opposite points on the boundary.[1]

Compute the singular homology of M'_g by using a CW model for M'_g similar to the one in the tutorial.

*Exercise 2: *Prescribing homology groups, coefficients*

(a) Construct a topological space X such that $H_n(X) = \mathbb{Z}/m\mathbb{Z}$ and $\widetilde{H}_i(X) = 0$ for all $i \neq n$.

(b) Given an abelian group G and a map of spheres $f\colon S^n \longrightarrow S^n$ of degree m. First argue that $\widetilde{H}_k(S^n; G)$ is zero for $k \neq n$ and $H_n(S^n; G) \cong G$. Then show that the induced map $f_*\colon H_n(S^n; G) \longrightarrow H_n(S^n; G)$ is given by multiplication by m. This allows us to define the notion of *degree* independently of coefficients.

(c) Construct topological spaces X and Y and a continuous map $f\colon X \longrightarrow Y$ such that f_* is trivial on homology with coefficients in $\mathbb{Z}$ while f_* is not trivial on homology with coefficients in $\mathbb{Z}/m\mathbb{Z}$. *Hint:* The cellular boundary formula from the tutorial holds for aribtrary coefficients by (b).

[1] See "Conway's ZIP proof" if you're curious about the classifcation of surfaces.

***Exercise 3:** *Tensor product, Tor functor, coefficients*

(a) Given integers m, n let (m, n) denote the greatest common divisor of m and n. Let $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$, likewise $\mathbb{Z}_n$. Show that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)}$.

(b) Given abelian groups G and H, show that $\mathrm{Tor}_1^{\mathbb{Z}}(H, G) = 0$ if H or G is torsionfree. *Hint:* You may use that $- \otimes_{\mathbb{Z}} G$ is an exact functor if G is torsionfree.

(c) Let G be an abelian group. Show that $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_m, G) \cong \ker(G \xrightarrow{m} G)$.

(d) Show that given finitely generated abelian groups A and B the group $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is isomorphic to the tensor product of finite cyclic groups. *Hint:* First compute $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n)$. You may use that $\mathbb{Z}/n\mathbb{Z} \cong m\mathbb{Z}/nm\mathbb{Z}$ (follows from an isomorphism theorem).

(e) Let K be a finite CW complex (or finite simplicial complex). Show that for every n, the n-th singular homology group is uniquely determined (up to isomorphism) by the i-th homology groups with coefficients in the elementary primary groups, that is, by the groups $H_i(K; \mathbb{Z}_q)$ for all $i \in [0, n]$ and all $q = p^k$ with p prime and $k \geq 0$.