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Topologie II – Exercise Sheet 8

Date of assignment: **Tuesday, Feb. 3, 2015**. We highly recommend problems marked with a star.

Exercise 1: *Properties of* $\mathrm{Hom}_{\mathbb{Z}}(-,G)$

Let A, B, G and $\{A_\alpha : \alpha \in J\}$ be abelian groups. Prove the following claims:

(a) $\mathrm{Hom}(\bigoplus_\alpha A_\alpha, G) \cong \prod_\alpha \mathrm{Hom}(A_\alpha, G)$.

(b) If A is finitely generated, then $\mathrm{Hom}(A,\mathbb{Z}) \cong A/T(A)$ where $T(A) \subset A$ is the torsion subgroup of A.

(c) If A is infinite cyclic, then $\mathrm{Hom}(A,G) \cong G$.

*Exercise 2: *Properties of* $\mathrm{ext}^n_{\mathbb{Z}}(-,G)$

Let A, B, G and $\{A_\alpha : \alpha \in J\}$ be abelian groups. Prove the following claims:

(a) $\mathrm{ext}^n_{\mathbb{Z}}(\bigoplus_\alpha A_\alpha, G) \cong \prod_\alpha \mathrm{ext}^n_{\mathbb{Z}}(A_\alpha, G)$.

(b) $\mathrm{ext}^n_{\mathbb{Z}}(A,G) = 0$ for $n > 1$.

(c) $\mathrm{ext}^1_{\mathbb{Z}}(A,G) = 0$ if A is free abelian.

(d) $\mathrm{ext}^0_{\mathbb{Z}}(A,G) = \mathrm{Hom}(A,G)$.

(e) If A is finitely generated, then $\mathrm{ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) = T(A)$, where $T(A) \subset A$ is the torsion subgroup of A.

*Exercise 3: *Cohomology*

(a) Let X be a topological space. Denote by $H^n(X;\mathbb{Z})$ its n-th cohomology group with coefficients in $\mathbb{Z}$. Show that if the singular homology groups $H_n(X,\mathbb{Z})$ and $H_{n-1}(X,\mathbb{Z})$ are finitely generated, then

$$H^n(X;\mathbb{Z}) \cong \big(H_n(C)/T(H_n(C))\big) \oplus T(H_{n-1}(X)).$$

Here $T(-)$ denotes the torsion subgroup.

(b) Let X be a topological space and let F be a field. Let $C_n(X;F)$ be the n-th singular chain group with coefficients in F. Let $\mathrm{Hom}_F(C_n(X;F),F)$ be the set of F-vector space homomorphisms from $C_n(X;F)$ to F. Show that $\mathrm{Hom}_F(C_n(X;F),F) \cong \mathrm{Hom}_{\mathbb{Z}}(C_n(X,\mathbb{Z}),F)$ as F-vector spaces. Use this to show that

$$H^n(X;F) \cong \mathrm{Hom}_F(H_n(X;F),F) \quad \text{(as } F\text{-vector spaces)}.$$

(c) Let G be an abelian group. Show that if $f\colon S^n \longrightarrow S^n$ has degree d, then the induced map $f^*\colon H^n(X;G) \longrightarrow H^n(X;G)$ is multiplication by d.

(d) Compute the cohomology of a non-oriented surface M'_g of genus g with coefficients in $\mathbb{Z}_2$ and in $\mathbb{Z}$. You may assume that the singular homology of M'_g with $\mathbb{Z}$ coefficients is known.