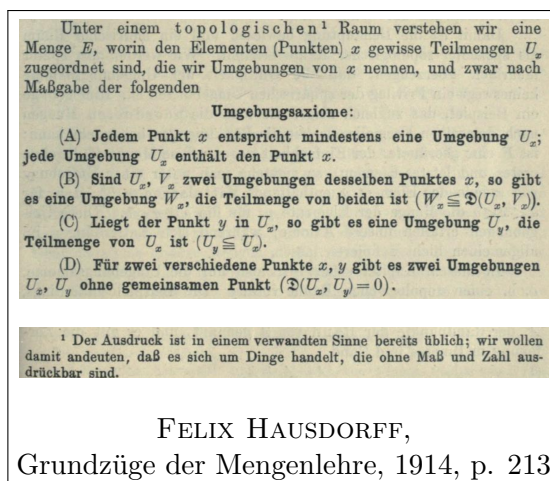


Exercise Sheet for *Topology I*, 2017/18

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Sheet 1

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Exercise 1 (Kuratowski closure axioms)

We consider a set X together with a map

$$\bar{\cdot} : \mathcal{P}(X) \rightarrow \mathcal{P}(X), \quad A \mapsto \bar{A}$$

such that for all $A, B \subset X$ we have

1. $\bar{\emptyset} = \emptyset$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
3. $A \subset \bar{A}$
4. $\overline{\bar{A}} = \bar{A}$.

Let $\mathcal{C} \subset \mathcal{P}$ be the collection of sets $C \subset X$, such that $\bar{C} = C$. Prove that (X, \mathcal{C}) is a topological space defined by a collection of closed sets \mathcal{C} .

Given a topological space defined by closed sets, how can one define a closure map that satisfies the axioms above?

Exercise 2 (New metrics from old ones)

Let (M, d) be a metric space. Let $d_1, d_2 : X \times X \rightarrow \mathbb{R}$ be defined as

$$d_1(x, y) = \min\{d(x, y), 1\} \quad \text{and} \quad d_2(x, y) = \frac{d(x, y)}{d(x, y) + 1}.$$

1. Show that d_1 defines a metric on M .
2. Show that d_2 defines a metric on M .
3. Show that d, d_1 and d_2 induce the same topology on M .

Exercise 3 (Infinite product of metric spaces)

Let $(M_i, d_i)_{i \in \mathbb{N}}$ be a family of metric spaces. Consider the space $M := \prod_{i \in \mathbb{N}} M_i$ together with the metric

$$d((x_0, x_1, \dots), (y_0, y_1, \dots)) := \sum_{i \in \mathbb{N}} 2^{-i} \frac{d(x_i, y_i)}{d(x_i, y_i) + 1}.$$

Show that (M, d) is indeed a metric space.

Exercise 4 (Finite topologies)

We consider topological spaces (X, \mathcal{O}) , where X is a finite set and $\mathcal{O} \subset \mathcal{P}(X)$ is the set of open subsets of X . Let $X = \{0, 1, 2, 3, 4\}$ be a set with 5 elements. Provide at least 6 different collections of open sets \mathcal{O} , such that (X, \mathcal{O}) is a topological space.

Exercise 5 (Fürstenbergs proof)

Consider the integers \mathbb{Z} . For $a, b \in \mathbb{Z}$ we define a subset $S(a, b) \subset \mathbb{Z}$ as

$$S(a, b) := a\mathbb{Z} + b = \{az + b \mid z \in \mathbb{Z}\}.$$

We call a subset $A \subset \mathbb{Z}$ *open* if it is either empty or a union of sets of the form $S(a, b)$ with $a \neq 0$. Show that

1. The collection of open sets defines a topology on \mathbb{Z} .
2. A nonempty finite subset of \mathbb{Z} cannot be open.
3. The set $\mathbb{Z} \setminus \{-1, 1\}$ cannot be closed.
4. The sets $S(a, b)$ are closed sets, i.e. can be written as complements of open sets.
5. When taking the union over all prime numbers p , we have $\bigcup_p S(p, 0) = \mathbb{Z} \setminus \{-1, 1\}$.
6. Use 5. to show that there are infinitely many primes.



FELIX HAUSDORFF at his desk, June 8.-14, 1924