Exercise Sheet for *Topology I*, 2017/18

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Sheet 1 due Wednesday, October 25th, 2017

Unter einem topologischen[1] Raum verstehen wir eine Menge E, worin den Elementen (Punkten) x gewisse Teilmengen U_x zugeordnet sind, die wir Umgebungen von x nennen, und zwar nach Maßgabe der folgenden

Umgebungsaxiome:

(A) **Jedem Punkt x entspricht mindestens eine Umgebung U_x; jede Umgebung U_x enthält den Punkt x.**

(B) **Sind U_x, V_x zwei Umgebungen desselben Punktes x, so gibt es eine Umgebung W_x, die Teilmenge von beiden ist ($W_x \leqq \mathfrak{D}(U_x, V_x)$).**

(C) **Liegt der Punkt y in U_x, so gibt es eine Umgebung U_y, die Teilmenge von U_x ist ($U_y \leqq U_x$).**

(D) **Für zwei verschiedene Punkte x, y gibt es zwei Umgebungen U_x, U_y ohne gemeinsamen Punkt ($\mathfrak{D}(U_x, U_y) = 0$).**

[1] Der Ausdruck ist in einem verwandten Sinne bereits üblich; wir wollen damit andeuten, daß es sich um Dinge handelt, die ohne Maß und Zahl ausdrückbar sind.

Felix Hausdorff,
Grundzüge der Mengenlehre, 1914, p. 213

Exercise 1 (Kuratowski closure axioms)

We consider a set X together with a map

$$\overline{\cdot}\colon \mathcal{P}(X) \to \mathcal{P}(X), \qquad A \mapsto \overline{A}$$

such that for all $A, B \subset X$ we have

1. $\overline{\varnothing} = \varnothing$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
3. $A \subset \overline{A}$
4. $\overline{\overline{A}} = \overline{A}$.

Let $\mathcal{C} \subset \mathcal{P}$ be the collection of sets $C \subset X$, such that $\overline{C} = C$. Prove that $(X, \mathcal{C})$ is a topological space defined by a collection of closed sets $\mathcal{C}$.

Given a topological space defined by closed sets, how can one define a closure map that satisfies the axioms above?

Exercise 2 (New metrics from old ones)

Let (M, d) be a metric space. Let $d_1, d_2\colon X \times X \to \mathbb{R}$ be defined as

$$d_1(x,y) = \min\{d(x,y), 1\} \quad \text{and} \quad d_2(x,y) = \frac{d(x,y)}{d(x,y)+1}.$$

1. Show that d_1 defines a metric on M.
2. Show that d_2 defines a metric on M.
3. Show that d, d_1 and d_2 induce the same topology on M.

Exercise 3 (Infinite product of metric spaces)

Let $(M_i, d_i)_{i\in\mathbb{N}}$ be a family of metric spaces. Consider the space $M := \prod_{i\in\mathbb{N}} M_i$ together with the metric

$$d((x_0, x_1, \dots), (y_0, y_1, \dots)) := \sum_{i\in\mathbb{N}} 2^{-i} \frac{d(x_i, y_i)}{d(x_i, y_i) + 1}.$$

Show that (M, d) is indeed a metric space.

Exercise 4 (Finite topologies)

We consider topological spaces $(X, \mathcal{O})$, where X is a finite set and $\mathcal{O} \subset \mathcal{P}(X)$ is the set of open subsets of X. Let $X = \{0, 1, 2, 3, 4\}$ be a set with 5 elements. Provide at least 6 different collections of open sets $\mathcal{O}$, such that $(X, \mathcal{O})$ is a topological space.

Exercise 5 (Fürstenbergs proof)

Consider the integers $\mathbb{Z}$. For $a, b \in \mathbb{Z}$ we define a subset $S(a, b) \subset \mathbb{Z}$ as

$$S(a, b) := a\mathbb{Z} + b = \{az + b \,|\, z \in \mathbb{Z}\}.$$

We call a subset $A \subset \mathbb{Z}$ *open* if it is either empty or a union of sets of the form $S(a, b)$ with $a \neq 0$. Show that

1. The collection of open sets defines a topology on $\mathbb{Z}$.
2. A nonempty finite subset of $\mathbb{Z}$ cannot be open.
3. The set $\mathbb{Z} \setminus \{-1, 1\}$ cannot be closed.
4. The sets $S(a, b)$ are closed sets, i.e. can be written as complements of open sets.
5. When taking the union over all prime numbers p, we have $\bigcup_p S(p, 0) = \mathbb{Z} \setminus \{-1, 1\}$.
6. Use 5. to show that there are infinitely many primes.

Felix Hausdorff at his desk, June 8.-14, 1924