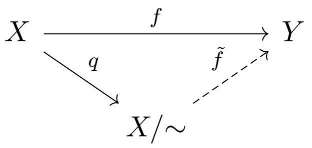
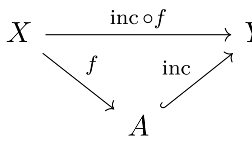
Exercise Sheet for *Topology I*, 2017/18

Prof. Pavle Blagojević, Dr. Moritz Firsching, Jonathan Kliem

Sheet 11 due Friday, January 26th, 2018



The universal property of the quotient: Suppose f is continuous. Then $\tilde{f}$ is continuous if it is well-defined (such that the diagram commutes).



The universal property of a subspace: Let $A \subset Y$ be a subspace with inclusion map $\mathrm{inc}\colon A \to Y$. Then f is continuous if and only if $\mathrm{inc} \circ f$ is continuous.

Exercise 41 (Deriving (Sub-)basisis from old ones)

Motivation Let X, Y be topological spaces and $f\colon X \to Y$ be some set map. If $\mathcal{S}$ is a subbasis of Y, then f is continuous if and only if $f^{-1}(U)$ is open in X for all $U \in \mathcal{S}$.

(Slogan: For continuity it is sufficient to check a subbasis.)

Let $\mathcal{B}$ be a basis of X and $S \subset X$. S is open if and only if for every $x \in S$ there exists some $U \in \mathcal{B}$ such that $x \in U \subset S$.

Overall, to prove continuity of $f\colon X \to Y$ it is usefull to have a subbasis of Y. To disprove continuity it is usefull to have a basis of X.

1. Product. Let X_1, X_2 be topological spaces with basis $\mathcal{B}_1, \mathcal{B}_2$. Prove that $\mathcal{B}_1 \times \mathcal{B}_2$ is a basis of $X_1 \times X_2$, where
$$\mathcal{B}_1 \times \mathcal{B}_2 := \{U_1 \times U_2 | U_1 \in \mathcal{B}_1, U_2 \in \mathcal{B}_2\}.$$
Let $(X_i)_{i \in I}$ be a family of topological spaces with $(\mathcal{B}_i)_{i \in I}$ be a family of corresponding basisis. Prove that $\prod_{i \in I} \mathcal{B}_i$ is a basis of $\prod_{i \in I} X_i$, where
$$\prod_{i \in I} \mathcal{B}_i := \{\prod_{i \in I} U_i | \forall i \in I\colon U_i \in \mathcal{B}_i, \text{ all but finitely many } U_i \text{ are equal to } X_i\}.$$

2. Coproduct. Let X_1, X_2 be topological spaces with subbasis $\mathcal{S}_1, \mathcal{S}_2$. Prove that $\mathcal{S}_1 \sqcup \mathcal{S}_2$ is a subbasis of $X_1 \sqcup X_2$, where
$$\mathcal{S}_1 \sqcup \mathcal{S}_2 := \{U | U \in \mathcal{S}_1 \sqcup \mathcal{S}_2.\}$$
Given a family of topological spaces $(X_i)_{i \in I}$ with subbasisis $(\mathcal{S}_i)_{i \in I}$. Construct a subbasis for $\bigsqcup_{i \in I} X_i$ from $(\mathcal{S}_i)_{i \in I}$.

3. Subspace. Let X be a topological space with basis $\mathcal{B}$ and let $A \subset X$ with the subspace topology. Construct a basis for A from $\mathcal{B}$.

4. Quotient space. Let X be topological space with subbasis $\mathcal{S}$ and let $X/\sim$ be the quotient obtained by some relation. Construct a subbasis for $X/\sim$ from $\mathcal{S}$.

Remark We have only asked for a basis or a subbasis for each construction. This is why:

1. To check wether a map $X \to Y_1 \times Y_2$ is continuous, one needs not to know a subbasis of $Y_1 \times Y_2$ but can instead check the map on each factor.
2. To check wether a map $X_1 \sqcup X_2 \to Y$ is continuous one can check each component.
3. To check wether a map $X \to A \subset Y$ is continuous one can check the induced map $X \to Y$.
4. To check wether a map $X/\sim \to Y$ is continuous one can check instead the induced map $X \to Y$.

However, if we have some construction $(X_1 \times X_2)/\sim$ then we might care about a subbasis for $X_1 \times X_2$ from a subbasis from X_1 and X_2 (which is possible by the same construction).

Exercise 42 (S^∞ is contractible) Consider the space

$$S^\infty := \Big\{(x_0, x_1, \dots)\Big| \sum_{i=0}^{\infty} x_i^2 = 1, \text{all but finitely many } x_i \text{ are zero}\Big\}.$$

This turns out to be a metric space by

$$d_2((x_0,\dots),(y_0,\dots)) := \sqrt{\sum_{i=0}^{\infty}(x_0 - y_0)^2}.$$

In this exercise we will show that this space is contractible.
Consider $S^n = \{(x_0,\dots,x_n)| \sum_{i=1}^n x_i^2 = 1\}$ with the Euclidean metric

$$d_2((x_0,\dots,x_n),(y_0,\dots,y_n) := \sqrt{\sum_{i=0}^{n}(x_i - y_i)^2}.$$

As we will see, this space is not contractible. However each inclusion $S^n \to S^{n+1}$ is homotopic to a constant map. We will take particular interest in the inclusions

$$i_n \colon S^n \to S^{n+1}, \quad (x_0,\dots,x_n) \mapsto (x_0,\dots,x_n,0)$$

and

$$j_n \colon S^n \to S^{n+1}, \quad (x_0,\dots,x_n) \mapsto (0,x_0,\dots,x_n).$$

1. Show that i_n is homotopic to j_n.
2. Show that j_n is homotopic $(x_0,\dots,x_n) \mapsto (1,0,0,\dots,0)$.
3. Show that $i_\infty = \mathrm{id} \colon S^\infty \to S^\infty$ is homotopic to
$$j_\infty \colon S^\infty \to S^\infty, (x_0,\dots) \mapsto (0,x_0,\dots).$$
4. Show that j_∞ is homotopic to a constant map.
5. Conclude that S^∞ is contractible.

Exercise 43 (Homotopy of pairs vs. Homotopy of quotients) Suppose $f,g \colon (X,A) \to (Y,B)$ are continuous maps of pairs.

1. Prove that they induce continuous maps $\tilde f, \tilde g \colon X/A \to Y/B$.
2. Suppose f and g are homotopic as maps of pairs. Prove that $\tilde f$ and $\tilde g$ are homotopic relative $[A]$, the point corresponding to A.
3. Suppose $\tilde f$ and $\tilde g$ are homotopic relative $[A]$. Give an example that f and g need not to be homotopic as maps of pairs. (You need not to prove your counterexample.)