Exercise Sheet for *Topology I*, 2017/18

Prof. Pavle Blagojević, Dr. Moritz Firsching, Jonathan Kliem

Sheet 12 due Friday, February 2nd, 2018

$\otimes$ $\otimes$

Theorem (Seifert, van Kampen) *Let $x \in X_0, X_1 \subset X$ such that $(X_0)^\circ \cup (X_1)^\circ = X$. Assume X_0, X_1 and $X_0 \cap X_1$ are path-connected, then the diagram*

$$\begin{array}{ccc} X_0 \cap X_1 & \hookrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_1 & \hookrightarrow & X \end{array}$$

induces a push-out diagram of groups

$$\begin{array}{ccc} \pi_1(X_0 \cap X_1, x) & \longrightarrow & \pi_1(X_0) \\ \downarrow & & \downarrow \\ \pi(X_1, x) & \longrightarrow & \pi_1(X, x) \end{array}$$

While our goal is to prove this theorem during the lectures, we will assume this to be true for the remainder of the exercises. This is very usefull to calculate fundamental groups, as we will see. (We already know, how to calculate a push-out of groups by prior exercises.)
Likewise we will assume that $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ is generated by $[f]$ for $f\colon I \to S^1$, $t \mapsto e^{2\pi t i}$.

Exercise 44 (Fundamental group of $S^1 \vee S^1$)

Consider $x = (1,0) \in S^1 \subset \mathbb{C} \cong \mathbb{R}^2$. We have seen that $X := (S^1, x) \vee (S^1, x)$ is given by

$$S^1 \times \{0,1\}/\sim, \quad (x,n) \sim (y,n') \Leftrightarrow x = y.$$

Define $X_i := X \backslash \{((1,0), i)\}$ for $i = 0,1$. This way each X_i is path-connected and contains x and we have $(X_0)^\circ \cup (X_1)^\circ = X$. So we are in a situation to apply Seifert-van Kampen.

Calculate $\pi_1(S^1 \vee S^1, x)$!

Exercise 45 (Putting a picture on the wall)

1. Using a rope, hang Seifert-van Kampens's Theorem in such a way that it is safely secured and such that removing any nail makes it fall to the ground.
2. How does one do this for three nails?
3. Let $x, y, z \in \mathbb{R}^2$ be pairwise disjoint. Find an element in $[f] \in \pi_1(\mathbb{R}^2 \backslash \{y,z\}, x)$ such that the maps induced by

$$\mathbb{R}^2 \backslash \{y,z\} \hookrightarrow \mathbb{R}^2 \backslash \{y\}, \quad \mathbb{R}^2 \backslash \{y,z\} \hookrightarrow \mathbb{R}^2 \backslash \{z\}$$

tirivialize $[f]$.

Exercise 46 (Fundamental group of $\bigvee_{i=1}^{n} S^1$)

Calculate $\pi_1\big(\bigvee_{i=1}^{n} S^1, x\big)$ where this is inductively defined by

$$\bigvee_{i=1}^{n}(S^1, x) = (S^1, x) \vee \Big(\bigvee_{i=1}^{n-1}(S^1, x), x\Big).$$

Exercise 47 (Classification of finite graphs up to homotopy) Show that every finite graph is homotopy equivalent to

$$\bigvee_{i=1}^{n_1} S^1 \sqcup \bigvee_{i=1}^{n_2} S^1 \sqcup \cdots \bigvee_{i=1}^{n_k} S^1$$

for some $k \in \mathbb{Z}$ and some $n_1 \leq n_2 \leq \cdots \leq n_k$, $n_i \in \mathbb{Z}_{\geq 0}$. Show that k and $n_1, \ldots, n_k$ do not depend on the choice of homotopy equivalence.

Exercise 48 (Fundamental group of S^n)

1. Use Seifert-van Kampen to calculate $\pi_1(S^2, x)$.
2. Conclude that $S^1 \hookrightarrow S^2$ is not a retract.
3. Use Seifert-van Kampen to calculate $\pi_1(S^n, x)$ for all $n \in \mathbb{N}$.

Exercise 49 (Stacking of Tori)

Compute the fundamental group of X. Here

$$X = (S^1 \times S^1) \sqcup (S^1 \times S^1)/\!\sim$$

where $\sim$ is generated by $(x, t, 0) \sim (x, t, 1)$ for fixed $x \in S^1$ and all $t \in S^1$.

Exercise 50 (Checking conditions of Seifert-van Kampen)

1. In Seifert and van-Kampen's Theorem we require $(X_0)^\circ \cup (X_1)^\circ = X$. Prove that one can not weaken this condition to $X_0 \cup X_1 = X$.
2. Prove that one can not weaken the theorem such that only X_0 and X_1 need to be path-connected.