Exercise Sheet for *Topology I*, 2017/18

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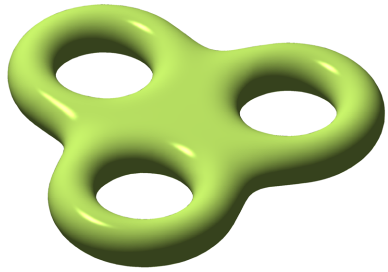
Sheet 13 due Friday, February 9th, 2018



An orientable surface of genus 3.

Exercise 51 (Klein bottle)

Calculate the fundamental group of the Klein bottle.

Exercise 52 (Calculate the fundamental group of the projective plane)

Calculate the fundamental group of $\mathbb{RP}^2$.

Exercise 53 (Fundamental group $X \vee Y$)

Let $x \in X$ be a path-connected topological space with an open neighborhood U of x that deformation-retracts to x.

Let $y \in Y$ be a path-connected topological space with an open neighborhood V of y that deformation-retracts to y.

Calculate $\pi_1\big((X,x)\vee(Y,y),x\big)$, where we identify $x \in X\vee Y$ with the image of x of the canonical inclusion $X \hookrightarrow X \vee Y$!

Exercise 54 (One circle in $\mathbb{R}^3$)

Consider

$$A = \{(x,y,z) \in \mathbb{R}^3 | z = 0,\ x^2 + y^2 = \tfrac{1}{2}\} \subset \mathbb{R}^3$$

the subspace of one circle.

We want to prove that $\mathbb{R}^3 \backslash A$ is homotopy equivalent to S^2 with one diameter attached.

1. Consider the map

   $$f\colon X = (\mathbb{R}_{\geq 0} \times [0,2\pi] \times [0,\pi])/\!\sim\ \to \mathbb{R}^3, \quad (r,u,v) \mapsto (r\cos(u)\sin(v), r\sin(u)\sin(v), r\cos(v)),$$

   where $\sim$ is generated by $(0,u,v) \sim (0,u',v')$ and $(r,0,v) \sim (r,2\pi,v)$. Prove that f is a homeomorphism.
2. Consider the subspace $B \subset X$ given by

   $$B = \{[(\tfrac{1}{2}, u, \tfrac{\pi}{2})] \in X.\}$$

   Prove that f induces a homeomorphism $X \backslash B \cong \mathbb{R}^3 \backslash A$.

3. Prove that $X \backslash B$ is homotopy equivalent to

$$C = \{[(r, u, v)] \in X | r = 1 \vee r = 0 \vee v = 0 \vee v = \pi\}.$$

(Recall Exercise 14.)

4. Conclude by using f that $X \backslash A$ is homeomorphic to

$$\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1 \vee x = y = 0\}.$$

Exercise 55 (Two unlinked circles in $\mathbb{R}^3$)

Consider

$$A = \{(x, y, z) \in \mathbb{R}^3 | z = \pm 1,\ x^2 + y^2 = \tfrac{1}{2}\} \subset \mathbb{R}^3$$

the subspace of two unlinked circles. Calculate $\pi_1(\mathbb{R}^3 \backslash A, 0)$!

Hint: It might be usefull to see that $\mathbb{R}^3 \backslash A$ is homotopy equivalent to $D^3 \vee D^3 \backslash A$. Where $D^2 \vee D^2$ is given by

$$\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + (z-1)^2 \leq 1 \vee x^2 + y^2 + (z+1)^2 \leq 1\}.$$

Now one can use the previous exercise. To find a good representative of $\mathbb{R}^3 \backslash A$.

Exercise 56 (Two linked circles in $\mathbb{R}^3$)

Consider

$$B = \{(x, y, z) \in \mathbb{R}^3 | (z = 0,\ x^2 + y^2 = 1) \vee (x = 0,\ y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

the subspace of two unlinked circles. Calculate $\pi_1(\mathbb{R}^3 \backslash B, 0)$!

Convince yourself that $\mathbb{R}^3 \backslash B$ is homotopy equivalent to $S^2 \vee (S^1 \times S^1)$. Having done that, you may use this, without a proof.

Remark: With this and the previous exercise, we have shown, that one can distinct two linked circles from two unlinked circles.

Exercise 57 (Lines through the origin in $\mathbb{R}^3$)

Let $X \subset \mathbb{R}^3$ be the union of n distinct lines through the origin. Calculate the fundamental group of $\mathbb{R}^3 \backslash X$.

Hint: Show first that $\mathbb{R}^3 \backslash X \simeq S^2 \backslash (X \cap S^2)$.

Exercise 58 (Fundamental group of the oriented surface of genus g)

The surface of genus g is obtained the following way. Take a regular $4g$-gon and identify the edges according to this formular:

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}.$$

As you can see the surface of genus 1 is the torus.

Calculate the fundamental group of the oriented surface of genus g. (Hint: Take the enlarged boundary of the polygon as X_1 and the interior as X_2 and then apply Seifert-van Kampen).

Why may we say **the** oriented surface of genus g?