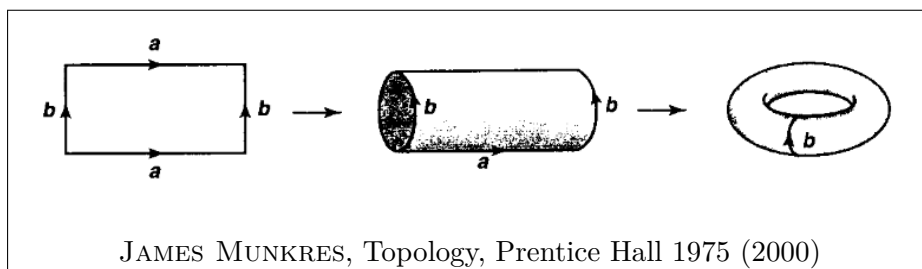


Exercise Sheet for *Topology I*, 2017/18

Prof. Pavle Blagojević, Dr. Moritz Firsching, Jonathan Kliem

Sheet 2

due Wednesday, November 1st, 2017



Exercise 6 (Norms on function spaces)

Consider the space $C([0, 1])$ of continuous, real functions $[0, 1] \rightarrow \mathbb{R}$. We can define the following metrics on $C([0, 1])$:

$$d_\infty(f, g) := \sup_x |f(x) - g(x)|$$
$$d_2(f, g) := \sqrt{\int_0^1 (f(x) - g(x))^2 dx}.$$

Show that both d_∞ and d_2 give a metric on $C([0, 1])$. The evaluation map is defined as

$$E: C([0, 1]) \rightarrow \mathbb{R}, \quad f \mapsto f(0).$$

Show that E is continuous when viewed as a map from the metric space $(C([0, 1]), d_\infty)$, but *not* continuous when viewed as a map from the metric space $(C([0, 1]), d_2)$. Hence the two metrics are not equivalent.

Exercise 7 (Distance from points and sets)

Let (M, d) be a metric space. Show that

- For all $a \in M$, the function $f: M \rightarrow \mathbb{R}, x \mapsto d(x, a)$ is continuous.
- For all non-empty subsets $A \subset M$, the function $g: M \rightarrow \mathbb{R}, x \mapsto \inf\{d(x, a) | a \in A\}$, is continuous and $g^{-1}(\{0\}) = \overline{A}$.
- For every non-empty, disjoint and closed subsets $A, B \subset M$, there is a continuous function $h: M \rightarrow \mathbb{R}$, such that $h^{-1}(\{0\}) = A$ and $h^{-1}(\{1\}) = B$.

Exercise 8 (Induced metric and induced topology)

Let X be some space and let (Y, d) be some metric space with topology induced by that metric. Given a map $f: X \rightarrow Y$.

- (a) **Let f be injective.** There is an induced metric on X given by $d_f(x, y) := d(f(x), f(y))$. Show that this is a metric.
- (b) There is also an induced topology on X given by the preimages of open sets (**even if f is not injective**):

$$\{f^{-1}(U), U \subset Y \text{ open}\}.$$

Show that this is a topology. (It is the coarsest topology on X such that the map is continuous.)

- (c) **Assume that f is injective.** Show that the topology induced by d_f is the topology induced by f as in (b).

Exercise 9 (Alternatives to continuity)

Let $f: X \rightarrow Y$ be a map between two topological spaces. Then the following are equivalent:

- (a) f is continuous;
- (b) For all $B \subset Y$, we have $f^{-1}(B^\circ) \subset f^{-1}(B)^\circ$;
- (c) For all $C \subset Y$, we have $\overline{f^{-1}(C)} \subset f^{-1}(\overline{C})$.

For the map $f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto e^{-x^2}$, find two sets B and C such that the above inclusions are proper inclusions (and not equalities of sets).