Exercise Sheet for *Topology I*, 2017/18

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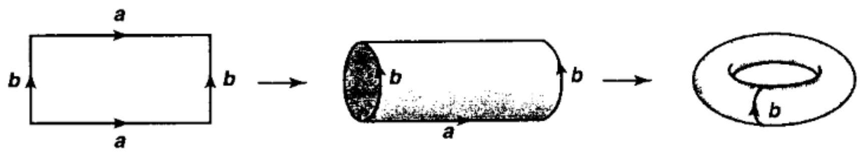
Sheet 2 due Wednesday, November 1st, 2017



James Munkres, Topology, Prentice Hall 1975 (2000)

Exercise 6 (Norms on function spaces)

Consider the space $C([0,1])$ of continuous, real functions $[0,1] \to \mathbb{R}$. We can define the following metrics on $C([0,1])$:

$$d_\infty(f,g) := \sup_x |f(x) - g(x)|$$
$$d_2(f,g) := \sqrt{\int_0^1 (f(x) - g(x))^2 dx}.$$

Show that both d_∞ and d_2 are give a metric on $C([0,1])$. The evaluation map is defined as

$$E\colon C([0,1]) \to \mathbb{R}, \qquad f \mapsto f(0).$$

Show that E is continuous when viewed as a map from the metric space $(C([0,1]), d_\infty)$, but *not* continuous when viewed as a map from the metric space $(C([0,1]), d_2)$. Hence the two metrics are not equivalent.

Exercise 7 (Distance from points and sets)

Let (M, d) be a metric space. Show that

(a) For all $a \in M$, the function $f\colon M \to \mathbb{R}, x \mapsto d(x,a)$ is continuous.

(b) For all non-empty subsets $A \subset M$, the function $g\colon M \to \mathbb{R}, x \mapsto \inf\{d(x,a) | a \in A\}$, is continuous and $g^{-1}(\{0\}) = \overline{A}$.

(c) For every non-empty, disjoint and closed subsets $A, B \subset M$, there is a continuous function $h\colon M \to \mathbb{R}$, such that $h^{-1}(\{0\}) = A$ and $h^{-1}(\{1\}) = B$.

Exercise 8 (Induced metric and induced topology)

Let X be some space and let (Y, d) be some metric space with topology induced by that metric. Given a map $f\colon X \to Y$.

- There is an induced metric on X given by $d_f(x, y) := d\left(f(x), f(y)\right)$. Show that this is a metric.
- There is also an induced topology on X given by the preimages of open sets:

  $$\{f^{-1}(U), U \subset Y \text{ open}\}.$$

  Show that this is a topology. (It is the coarsest topology on X such that the map is continuous.)
- Show that the topology induced by d_f is the topology induced by f as in (ii).

Exercise 9 (Alternatives to continuity)

Let $f\colon X \to Y$ be a map between two topological spaces. Then the following are equivalent:

(a) f is continuous;

(b) For all $B \subset Y$, we have $f^{-1}(B^\circ) \subset f^{-1}(B)^\circ$;

(c) For all $C \subset Y$, we have $\overline{f^{-1}(C)} \subset f^{-1}(\overline{C})$.

For the map $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto e^{-x^2}$, find two sets B and C such that the above inclusions are proper inclusions (and not equalities of sets).