Exercise Sheet for *Topology I*, 2017/18

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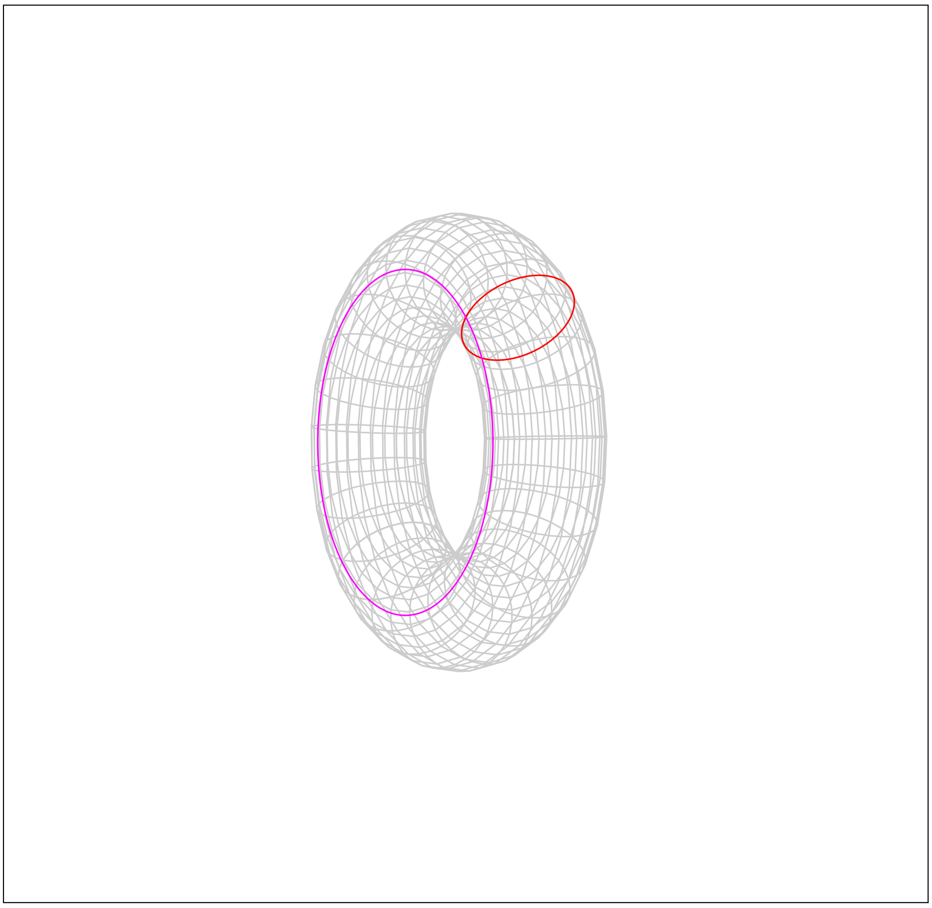
Sheet 4 due Wednesday, November 15th, 2017



Exercise 14 (Flat tire)

Consider the torus $T = S^1 \times S^1$ with the point $((-1,0),(-1,0))$ removed

$$X := S^1 \times S^1 \setminus \{((-1,0),(-1,0))\}.$$

This space is called a *punctured torus.* Now consider the subspace $Y \subset X$,

$$Y := S^1 \times \{(0,1)\} \cup \{(1,0)\} \times S^1.$$

Show that Y is a deformation retract of X.

Exercise 15 (Homotopic to a constant map)

Given a continous map $f\colon S^n \to X$, from the n-dimensional sphere to a space X, show that the following statements are equivalent.

(a) f is homotopic to a constant map.

(b) There is continous map $g\colon D^{n+1} \to X$, such that $g_{S^n} = f$, where we identify S^n as a subset of the $n+1$-dimensional closed ball D^{n+1}.

(c) There is continous map $h\colon S^{n+1} \to X$, such that $h_{S^n} = f$, where we identify S^n as a subset (equator) of the $n+1$-dimensional sphere S^{d+1}.

Exercise 16 (Homotopy Extension Property)

Given a space X and a subspace $A \subset X$. We say that the pair (X, A) has the *homotopy extension property* if for every $F\colon X \to Y$ and every homotopy

$$h\colon A \times [0,1] \to Y,$$

between F_A and some $g\colon A \to Y$ there is a homotopy

$$H\colon X \times [0,1] \to Y,$$

which restricts to h, i.e. $H_{A\times[0,1]} = h$.

Show that the following statements are equivalent.

(a) (X, A) has the homotopy extension property.

(b) $(X \times \{0\} \cup A \times I)$ is a deformation retract of $X \times I$.

Exercise 17 (Quotient of the torus)

Consider the torus $T := S^1 \times S^1$ and consider the following equivalence relation on T:

$$(x, y) \sim (y, x).$$

Let X be the quotient $T/\sim$. Define the Möbius strip to be the space $M := [-1,1] \times [-1,1]/\approx$, where $\approx$ is the equivalence relation $(x, 1) \approx (-x, -1)$. Show that X is homeomorphic to M. To what subset of the torus are the diagonal elements $\{[(x, x)] | (x, x) \in T\} \subset X$ mapped under a homeomorphism between X and M.