Exercise Sheet for *Topology I*, 2017/18

Prof. Pavle Blagojević, Dr. Moritz Firsching, Jonathan Kliem

Sheet 5 due Wednesday, November 22th, 2017

$$\begin{tikzcd}
 & Z \arrow[d, dashed, "\exists!" ', "F"] \arrow[ddl, "f_1"'] \arrow[ddr, "f_2"] & \\
 & X_1 \times X_2 \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

The universal property of the product:
Given topological spaces X_1, X_2. The product constists of a space $X_1 \times X_2$ and maps $p_i \colon X_1 \times X_2 \to X_i$, such that for every space Z together with maps $f_i \colon Z \to X_i$ there exists a unique map $Z \to X_1 \times X_2$ that the above commutes, i.e. $p_i \circ F = f_i$.
The universal property for groups is stated in the same way, just replace the spaces by groups and continuous maps by group homomorphisms. The product can be thought of as the smallest cone above X_1 and X_2.

Exercise 18 (Mapping Spaces)

1. Show that for any space X the space $[X, I]$ has exactly one element ($I = [0, 1]$).
2. Show that if Y is path-connected, then $[I, Y]$ has a single element.
3. Show that a contractible space is path-connected.
4. Show that if Y is contractible then for any X, $[X, Y]$ has a single element.
5. Show that if X is contractible and Y is path connected, then $[X, Y]$ has exactly one element.

Exercise 19 (Fundamental group of the product) Let (X, x_0), (Y, y_0) be spaces with base points. Given the product with the projection maps $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ there exist maps

$$\pi_1(p_1) \colon \pi_1(X \times Y) \to \pi_1(X)$$

and

$$\pi_1(p_2) \colon \pi_1(X \times Y) \to \pi_1(Y).$$

By the universal property of the product, there exists a unique map

$$\pi_1(p_1) \times \pi_1(p_2) \colon \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y).$$

Show that this map is an isomorphism. Use the universal property of the product $X \times Y$.

Hence we know e.g. that the fundamental group of the torus needs to be $\pi_1(S^1, x_0) \times \pi_1(S^1, x_0)$.

Exercise 20 (Fundamental group and retracts)

1. Proof that for a retract A of X the inclusion map $i\colon A \hookrightarrow X$ induces an injective group homomorphism $\pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0)$ for all $x_0 \in A$.
2. Proof that in for A being a deformation retract of X the inclusion induces an isomorphism of fundamental groups.

Exercise 21 (Some first applications of the fundamental group) Assuming that $\pi_1(S^1) \cong \mathbb{Z}$. Prove or disprove that the following are retracts/deformation retracts:

1. $S^1 \hookrightarrow \mathbb{R}^n$.
2. $i\colon S^1 \hookrightarrow S^1 \vee D^2$, where $S^1 \vee D^2$ (the wedge of S^1 and D^2) is obtained by gluing together disk and circle at a base point $x_0 \in S^1 \subset D^2$:

   $$S^1 \vee D^2 := S^1 \times \{0\} \cup D^2 \times \{1\}/(x_0, 0) \sim (x_0, 1).$$

   The inclusion $i\colon S^1 \to S^1 \vee D^2$ is given by $i(x) = (x, 0)$.
   How about $i(x) = (x, 1)$?
3. A point and S^1.
4. S^1 as the equator of S^2.
5. The circle S^1 and the Möbius band $D^1 \times D^1/\sim$ with $(0, t) \sim (1, 1-t)$ for all $t \in [0, 1]$. Here S^1 is viewed as $[0, 1] \times \{\frac{1}{2}\}/(0, \frac{1}{2}) \sim (1, \frac{1}{2})$.
6. The torus $S^1 \times S^1$ and the solid torus $S^1 \times D^2$.
7. Let $x \neq y$ be points in $\mathbb{R}^2$. $\{x, y\} \hookrightarrow \mathbb{R}^2$.
8. D^2 as the upper hemisphere of S^2.